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# The optimality condition of the multiple-cycle smoothed curve signal timing model

Li Li *, Kaidi Yang, Zhiheng Li, Zuo Zhang

*Department of Automation, Tsinghua University, Beijing 100084, China*

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ABSTRACT

In this paper, we study the global optimality of the multiple-cycle smoothed curve signal timing model that was proposed in Liu et al. (2008) for individual oversaturated intersections. First, we propose a counterexample to show that the recursively decomposing design method in Liu et al. (2008) cannot guarantee the global optimality of multiple-cycle signal timing plan for all traffic scenarios. Second, we give a simple sufficient global optimality condition and show that this condition will usually be satisfied in ordinary traffic scenarios. It indicates that the powerful smoothed curve signal timing model in Liu et al. (2008) is useful in practices. This finding also provides a good starting point to further analyze the performance of oversaturated intersections.



## 1. Introduction

If we consider the timing plan for several consequent signal cycles for an oversaturated intersection, the objective function often becomes very complex, because of the non-linearity of delay calculation formula. To conquer this problem, a creative method was proposed in Liu et al. (2008), which approximates the total delay by assuming a linear departure curve in every signal cycle. Then, we can decompose the multiple-cycle signal timing problem (MCP) into a sequence of single-cycle signal timing problem (SCP). Each SCP can be treated as a linear program that only depends on the remaining queue lengths of its previous cycle and solved recursively via any mature LP algorithms. This decomposition strategy greatly reduces the calculation costs and thus attracts increasing interests.

In Liu et al. (2008), some numerical testing results for this method were given. However, the global optimality of this recursive method had not been rigorously proven. In this paper, we will show that the original recursively decomposing design model cannot guarantee the global optimality of multiple-cycle signal timing plan, if no additional condition is added. Moreover, we will give a sufficient global optimality condition and show that this condition will usually be satisfied in ordinary traffic scenarios. This indicates that the smoothed curve model proposed in Liu et al. (2008) are useful in practical applications.

## 2. Problem presentation

### 2.1. The smoothed departure curve model

In this section, we briefly review the MCP model in Liu et al. (2008). For presentation convenience, we will adopt the symbols and notations of Zhao et al. (2011) in the rest of this paper. The nomenclature list is as follows (see Table 1).

* Corresponding author. Tel.: +86 10 62782071.
*E-mail address:* li-li@mail.tsinghua.edu.cn (L. Li).

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In MCP model, the total delay in oversaturated period can be calculated by the area between the piecewise-linear cumulative departure and arrival curve. The total delay in the first $n$ cycles of oversaturated period is written as

$$\text{Total Delay} = \sum_{p\in P}\sum_{m\in M^p}\sum_{k=1}^{n}(2n-2k+1)(\lambda_m(k)-\mu_m(k)) \tag{1}$$

Since cycle $C$ and arrival rate $\lambda_m(k)$ are known, the objective function of MCP is equal to

$$\min F := -\sum_{p\in P}\sum_{m\in M^p}\sum_{k=1}^{n}(2n-2k+1)\mu_m(k) \tag{2}$$

According to Zhao et al. (2011), the constraints of MCP can be listed as follows.

Since in any given cycle, the sum of green time ratios of all phases do not exceed the total allowable green time ratio, we have

$$\sum_{p\in P}\eta^p(k) \leqslant \eta, \quad \forall k = 1,\ldots,n \tag{3}$$

Considering lost time, the green time in each cycle should be no less than the minimum green time, i.e.

$$\eta^p(k)C \geqslant g^p_{\min}, \quad \forall p\in P,\ \forall k = 1,\ldots,n \tag{4}$$

Note that the departure flow rate is bounded by intersection capacity and the number of vehicles that arrival, we have

$$\mu_m(k) = \min\{\eta^p(k)s^p_m, X_m(k)/C + \lambda_m(k)\}, \quad \forall m\in M,\ \forall k = 1,\ldots,n \tag{5}$$

For the queue lengths in two consecutive cycles, we have

$$X_m(k+1) = X_m(k) + \lambda_m(k)C - \mu_m(k)C, \quad \forall m\in M,\ \forall k = 1,\ldots,n \tag{6}$$

In Liu et al. (2008), the linear programming model Eqs. (2)–(6) were solved recursively, because usually we do not know the numbers of arriving vehicles in the future cycles. In other words, we will first solve the sub linear programming model for the first cycle with the initial condition. Then, we get the control plan as the solution and the resulting remaining queue

**Table 1**
Nomenclature list.

| | |
|---|---|
| *The symbols below are treated as constants* | |
| $M$ | The set of traffic streams, indexed by $m$ |
| $P$ | The set of phases, indexed by $p$ |
| $L$ | The number of phases |
| $M^p$ | The set of allowable traffic streams in phase $p$ |
| $N^p$ | The number of allowable traffic streams in phase $p$ |
| $n$ | The number of signal cycles in over-saturated period |
| $g^p_{\min}$ | The minimum allowable green time in phase $p$ |
| $C$ | The length of a signal cycle |
| $s^p_m$ | The saturated departure flow rate for traffic stream $m$ in phase $p$ |
| $\eta$ | The total allowable green time ratio |
| $\lambda_m(k)$ | The arrival flow rate for traffic stream $m$ in signal cycle $k$ |
| *The symbols below are variables derived from above variables* | |
| $\eta^p(k)$ | The green time ratio of phase $p$ in cycle $k$, i.e, $\eta^p(k) = g^p(k)/C$ |
| $\eta(k)$ | The green time ratio vector for all phases in cycle $k$, i.e. $\eta(k) = (\eta^1(k), \eta^2(k), \ldots, \eta^L(k))$ |
| $\hat{\eta}^p(k)$ | The solution to MCP obtained by recursive greedy recursive; |
| $\mu_m(k)$ | The smoothed departure flow rate in cycle $k$ in traffic stream $m$ |
| $X_m(k)$ | The vehicle queue length of traffic stream $m$ in the beginning of signal cycle $k$ |
| $b^p_m(k)$ | The vehicle clean-up time for traffic stream $m$ in cycle $k$ under saturated flow rate, more precisely, $b^p_m(k) = (X_m(k)/C + \lambda_m(k))/s^p_m$ |
| $d^p_m(k)$ | The clean-up time for traffic stream $m$ that arrive in the beginning of cycle $k$ under saturated flow rate, more precisely, $d^p_m(k) = \lambda_m(k)/s^p_m$ |
| *The symbols below are functions and sets derived from above variables* | |
| $F$ | The objective function of MCP, equivalent to total delay |
| $h_k$, $\psi_{p,k}$ | The constraints of MCP |
| $F_k$ | The negative total departure flow rate of the first $k$ cycles, i.e. $F_k := \sum_{p\in P}\sum_{m\in M^p}\sum_{i=1}^{k}\max\{-\eta^p(i)s^p_m, -X(i)/C - \lambda(i)\}$ |
| $\partial F_k$ | The subdifferential of $F_k$ at the solution found by greedy search |
| $f^p_m(k)$ | The negative total departure flow rate in phase $p$ and stream $m$ in the first $k$ cycle, i.e. $-\sum_{i=1}^{k}\mu_m(i) = \sum_{i=1}^{k}\max\{-\eta^p(i)s^p_m, -\lambda_m(i) - X_m(i)/C\}$ |
| $\partial f^p_m(k)$ | The subdifferential of $f^p_m(k)$ at the solution found by greedy search |
| $\overline{\text{co}}(\Omega)$ | The closed convex hull of set $\Omega$, i.e. $\overline{\text{co}}(\Omega) = \{\theta_1\omega_1 + \theta_2\omega_2 + \cdots + \theta_n\omega_n \mid \omega_i \in \Omega, \sum_{i=1}^{n}\theta_i = 1\}$ |
| $\lfloor x \rfloor$ | The maximum integer less than $x \in \mathcal{R}$ |
| $\lceil x \rceil$ | The minimum integer greater than $x \in \mathcal{R}$ |