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Generic polar harmonic transforms for invariant image representation ☆

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ABSTRACT

This paper introduces four classes of rotation-invariant orthogonal moments by generalizing four existing moments that use harmonic functions in their radial kernels. Members of these classes share beneficial properties for image representation and pattern recognition like orthogonality and rotation-invariance. The kernel sets of these generic harmonic function-based moments are complete in the Hilbert space of square-integrable continuous complex-valued functions. Due to their resemble definition, the computation of these kernels maintains the simplicity and numerical stability of existing harmonic function-based moments. In addition, each member of one of these classes has distinctive properties that depend on the value of a parameter, making it more suitable for some particular applications. Comparison with existing orthogonal moments defined based on Jacobi polynomials and eigenfunctions has been carried out and experimental results show the effectiveness of these classes of moments in terms of representation capability and discrimination power.

1. Introduction

Rotation-invariant features of images are usually extracted by using moment methods [1] where an image f on the unit disk ($x^2+y^2 \le 1$) is decomposed into a set of kernels $\{V_{nm}|(n,m)\in\mathbb{Z}^2\}$ as

$$H_{nm} = \iint\limits_{x^2+y^2\le 1} f(x,y)V^*_{nm}(x,y)dxdy,$$

where the asterisk denotes the complex conjugate. According to [2], a kernel that is "invariant in form" with respect to rotation about the origin must be defined as

$$V_{nm}(r,\theta) = R_n(r)A_m(\theta),$$

where $r=\sqrt{x^2+y^2}, \theta = \mathrm{atan2}(y,x), A_m(\theta)=\mathrm{e}^{im\theta}$, and R_n could be of any form. For example, rotational moments (RM) [3] and complex moments (CM) [4] are defined by using $R_n(r)=r^n$; continuous generic Fourier descriptor (GFD) [5] employs $\mathrm{e}^{i2\pi nr}$ for $R_n(r)$; and angular radial transform (ART) [6] uses harmonic functions as

$$R_n(r) = \begin{cases} 1, & n=0 \\ \cos(\pi nr), & n\neq 0. \end{cases}$$

However, the obtained kernels V_{nm} of RM, CM, GFD, and ART are not orthogonal and, as a result, information redundancy exists in the moments H_{nm}, leading to difficulties in image reconstruction and low accuracy in pattern recognition, etc. Undoubtedly, orthogonality between kernels V_{nm} comes as a natural solution to these problems. Orthogonality means

$$\langle V_{nm}, V_{n'm'}\rangle = \iint\limits_{x^2+y^2\le 1} V_{nm}(x,y)V^*_{n'm'}(x,y)dxdy$$
$$= \int_0^1 R_n(r)R^*_{n'}(r)rdr\int_0^{2\pi}A_m(\theta)A^*_{m'}(\theta)d\theta = \delta_{nn'}\delta_{mm'},$$

where $\delta_{ij}=[i=j]$ is the Kronecker delta function. It can be seen from the orthogonality between the angular kernels

$$\int_0^{2\pi}A_m(\theta)A^*_{m'}(\theta)d\theta = \int_0^{2\pi}\mathrm{e}^{im\theta}\mathrm{e}^{-im'\theta}d\theta = 2\pi\delta_{mm'}$$

that the remaining condition on the radial kernels is

$$\int_0^1 R_n(r)R^*_{n'}(r)rdr = \frac{1}{2\pi}\delta_{nn'}. \tag{1}$$

This equation presents the regulating condition for the definition of a set of radial kernels R_n in order to have orthogonality between kernels V_{nm}.

There exists a number of methods that have their radial kernels satisfying the condition in Eq. (1) and they can be roughly classified into three groups. The first employs *Jacobi polynomials* [7] in r of order n for $R_n(r)$ obtained by orthogonalizing sequences of polynomial functions or by directly using existing orthogonal polynomials. Members of this group are Zernike moments (ZM) [8], pseudo-Zernike moments (PZM) [2], orthogonal Fourier–Mellin moments (OFMM) [9], Chebyshev–Fourier moments (CHFM) [10], and pseudo Jacobi–Fourier moments (PJFM)

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[11] (see [12, Section 6.3], or [13, Section 3.1] for a comprehensive survey). It was demonstrated recently that the Jacobi polynomial-based radial kernels of these methods are special cases of the shifted Jacobi polynomials [14,15]. Despite its popularity, this group of orthogonal moments however involves computation of factorial terms, resulting in high computational complexity and numerical instability, which often limit their practical usefulness.

The second group employs the *eigenfunctions* of the Laplacian ∇^2 on the unit disk as V_{nm}, similar to the interpretation of Fourier basis as the set of eigenfunctions of ∇^2 on a rectangular domain. These eigenfunctions are obtained by solving the Helmholtz equation, $\nabla^2 V + \lambda^2 V = 0$, in polar coordinates to have the radial kernels defined based on the Bessel functions of the first and second kinds [16]. In addition, by imposing the condition in Eq. (1) a class of orthogonal moments is obtained [17] and different boundary conditions were used for the proposal of a number of methods with distinct definition of λ: Fourier–Bessel modes (FBM) [18], Bessel–Fourier moments (BFM) [19], and disk-harmonic coefficients (DHC) [20]. However, the main disadvantage of these eigenfunction-based methods is the lack of an explicit definition of their radial kernels other than Bessel functions, leading to inefficiency in terms of computation complexity.

And the last group uses *harmonic functions* (i.e., complex exponential and trigonometric functions) for R_n by taking advantage of their orthogonality:

$$\int_0^1 e^{i2\pi nr} e^{-i2\pi n'r} dr = \delta_{nn'}, \tag{2}$$

$$\int_0^1 \cos(\pi nr)\cos(\pi n'r) dr = \frac{1}{2}\delta_{nn'}, \tag{3}$$

$$\int_0^1 \sin(\pi nr)\sin(\pi n'r) dr = \frac{1}{2}\delta_{nn'}, \tag{4}$$

$$\int_0^1 \cos(\pi nr)\sin(\pi n'r) dr = 0, \quad n - n' \text{ is even}. \tag{5}$$

It can be seen that the integrand in Eqs. (2)–(5) is "similar in form" with that in Eq. (1), except for the absence of the weighting term r which prevents a direct application of harmonic functions as radial kernels. This obstacle was first overcome in [21] by using the multiplicative factor $\frac{1}{\sqrt{r}}$ in the radial kernels to eliminate r in the definition of radial harmonic Fourier moments (RHFM) as

$$R_n(r) = \frac{1}{\sqrt{r}} \begin{cases} 1, & n = 0 \\ \sqrt{2}\sin(\pi(n+1)r), & n > 0 \ \& \ n \text{ is odd} \\ \sqrt{2}\cos(\pi nr), & n > 0 \ \& \ n \text{ is even}. \end{cases} \tag{6}$$

Recently, a different strategy was proposed to move r into the variable of integration, $r\mathrm{d}r = \frac{1}{2}\mathrm{d}r^2$, in the definition of three different forms of polar harmonic transforms [22]: polar complex exponential transform (PCET), polar cosine transform (PCT), and polar sine transform (PST). The radial kernels of these transforms are respectively defined as

$$R_n(r) = e^{i2\pi nr^2}, \tag{7}$$

$$R_n^C(r) = \begin{cases} 1, & n = 0 \\ \sqrt{2}\cos\left(\pi nr^2\right), & n > 0 \end{cases} \tag{8}$$

$$R_n^S(r) = \sqrt{2}\sin\left(\pi nr^2\right), \quad n > 0 \tag{9}$$

It is straightforward that the radial kernels in Eqs. (6)–(9) all satisfy the orthogonality condition in Eq. (1) and that their corresponding kernels are orthogonal over the unit disk. In addition, the radial kernels of RHFM in Eq. (6) are actually equivalent to $R_n(r) = \frac{1}{\sqrt{r}} e^{i2\pi nr}$ in terms of image representation, similar to the equivalence between different forms of Fourier series (namely trigonometric and complex exponential functions). The resemblance between the exponential form of RHFM's radial kernels and PCET's radial kernels suggests that they are actually special cases of a generic class of radial kernels that are defined based on complex exponential functions. And each member of this class can be used to define kernels that are orthogonal over the unit disk. Similar observation also leads to generic classes of radial kernels defined based on trigonometric functions.

The main contribution of this paper is a generic view on strategies that were used to define orthogonal moments. This leads to the introduction of four classes of radial kernels that correspond to four generic sets of moments and take existing harmonic moments as special cases. This paper proves theoretically that the generic sets of kernels are complete in the Hilbert space of all square-integrable continuous complex-valued functions over the unit disk. It also shows experimentally that the proposed harmonic moments are superior to Jacobi polynomial-based moments and are comparable to eigenfunction-based moments in terms of representation capability and discrimination power. It is also interesting to note that these generic harmonic moments can be computed very quickly by exploiting the recurrence relations among complex exponentials and trigonometric functions [23]. The generalization by introducing a parameter in this paper is similar to the generalization of the R-transform published recently [24].

The content of this paper is a comprehensive extension of the research work presented previously in [25]. The next section will derive explicit form of generic classes of radial kernels defined based on complex exponentials and trigonometric functions. The completeness of the sets of orthogonal decomposing kernels is proven in Section 3, along with some beneficial properties obtained from the generalization. Section 4 is devoted to the stability of the numerical computation. Experimental results in terms of representation capability and discrimination power are given in Section 5. And conclusions are finally drawn in Section 6.

2. Generic polar harmonic transforms

In order to formulate the generalization, assuming that the harmonic radial kernels have the generic exponential form $R_{ns}(r) = \kappa(r)e^{i2\pi nr^s}$, where $s \in \mathbb{R}$ and κ is a real functional of r. Then

$$\int_0^1 R_{ns}(r)R^*_{n's}(r) r dr = \int_0^1 \kappa^2(r) e^{i2\pi nr^s} e^{-i2\pi n'r^s} r \ dr.$$

Since $\mathrm{d}r^s = sr^{s-1}\mathrm{d}r = sr^{s-2}r\mathrm{d}r$ then

$$\int_0^1 R_{ns}(r)R^*_{n's}(r) r dr = \int_0^1 \frac{\kappa^2(r)}{sr^{s-2}} e^{i2\pi nr^s} e^{-i2\pi n'r^s} dr^s.$$

By letting $\frac{\kappa^2(r)}{sr^{s-2}} = \text{const} = C$,

$$\int_0^1 R_{ns}(r)R^*_{n's}(r) r dr = \int_0^1 C e^{i2\pi nr^s} e^{-i2\pi n'r^s} dr^s = C\delta_{nn'}.$$

In order to have orthonormality between kernels, it follows directly from Eq. (1) that $C = \frac{1}{2\pi}$. Then $\kappa(r) = \sqrt{\frac{sr^{s-2}}{2\pi}}$ and R_{ns} have the following actual definition:

$$R_{ns}(r) = \kappa(r) e^{i2\pi nr^s}, \tag{10}$$