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Inverses and quotients of mappings between ordered sets

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A B S T R A C T

In this paper, we study inverses and quotients of mappings between ordered sets, in particular between complete lattices, which are analogous to inverses and quotients of positive numbers. We investigate to what extent a generalized inverse can serve as a left inverse and as a right inverse, and how an inverse of an inverse relates to the identity mapping. The generalized inverses and quotients are then used to create a convenient formalism for dilations and erosions as well as for cleistomorphisms (closure operators) and anoiktomorphisms (kernel operators).

1. Introduction

Lattice theory is a mature mathematical theory thanks to the pioneering work by Garrett Birkhoff, Øystein Ore and others in the first half of the twentieth century. A standard reference is still Birkhoff's book [3], first published in 1940. Developments in lattice theory originate in several branches of mathematics, for instance algebra [6,5], logic [42,15], general topology and functional analysis [15, p.xxx–xxxii], convexity theory [40] and, most important as a background for this paper, mathematical morphology with applications in image processing (books by Matheron [28], Serra [35,38], and Heijmans [17]; articles by Heijmans and Ronse [20], Ronse [32], Ronse and Heijmans [33,34], and Serra [39]). Other areas where concepts from lattice theory are used include semantics (abstract interpretation) of programming, the theory of fuzzy sets, fuzzy logic, and formal concept analysis [13]. For general lattice theory a standard reference is Grätzer [16].

This variety of sources for fundamental concepts has led to varying terminology and hence to difficulties in tracing history.

In this paper, we shall study inverses and quotients of mappings between ordered sets which are analogous to inverses $1/y$ and quotients x/y of positive numbers. The theory of lower and upper inverses defined in Section 3 generalizes the theory of Galois connections as well as residuation theory and the theory of adjunctions. We investigate in Section 6 to what extent a generalized inverse can serve as a left inverse and as a right inverse, and how an inverse of an inverse relates to the identity mapping. The generalized inverses and quotients are then used in Section 9 to create a convenient formalism for a unified treatment of dilations $\delta : L \to M$ and erosions $\varepsilon : L \to M$ as well as of cleistomorphisms (closure operators) $\kappa : L \to L$ and anoiktomorphisms (kernel operators) $\alpha : L \to L$.

Often we require of the ordered sets studied that they shall be complete lattices. However, of the various phenomena brought together here, the Galois connections are the oldest, and they make

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sense between preordered sets which are not necessarily complete lattices or even lattices. A goal will therefore be to study this general situation and relate it to the more special theories of residuation and adjunction.

Both inverses and quotients come in two versions, lower and upper. It turns out that anoiktomorphisms can be characterized as lower quotients of the form $f/_\star f$, and cleistomorphisms as upper quotients $f/^\star f$.

To define an inverse of a general mapping seems to be a hopeless task. However, if the mapping is between preordered sets, there is some hope of constructing mappings that can serve in certain contexts just like inverses do. This is our task here.

Part of the results of the present paper were reported in my conference contribution (2007), however without proofs and with fewer examples. My lectures in the Spring Semesters of 2002 and 2004 also contained some of the results; see (2002a).

2. Definitions

Definition 2.1. A *preorder* in a set X is a binary relation which is reflexive (for all $x \in X, x \leqslant x$) and transitive (for all $x,y,z \in X$, $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$). An *order* is a preorder which is antisymmetric (for all $x,y \in X, x \leqslant y$ and $y \leqslant x$ imply $x = y$).

Birkhoff [2, p. 7] uses *quasi-ordered system*; in (1995:20) *quasi-ordering* and *quasi-ordered set*. Other terms are *preordering*, *quasiorder*, and *pseudoordering*. Nowadays *preorder* is more common (e.g., Gierz et al. [15, p. 1]).

To any preorder $\leqslant$ we introduce an equivalence relation $x \sim y$ defined as $x \leqslant y$ and $y \leqslant x$. If $\leqslant$ is an order, this equivalence relation is just equality. If we have two preorders, we say that $\leqslant_1$ is *stronger than* or *finer than* $\leqslant_2$ if for all x and y, $x \leqslant_1 y$ implies $x \leqslant_2 y$. We also say that $\leqslant_2$ is *weaker than* or *coarser than* $\leqslant_1$. The finest preorder is the *discrete order*, defined as equality; the coarsest preorder is the *chaotic preorder* given by $x \leqslant y$ for all x and y.

Definition 2.2. Given a preordered set $(X, \leqslant)$, we define the *opposite preordered set* $(X, \geqslant)$ as the set X equipped with the opposite preorder. We shall write X^{op} for this preordered set. Thus $x \leqslant_{X^{\mathrm{op}}} y$ if and only if $y \leqslant_X x$.

Definition 2.3. Given a mapping $f : X \to (Y, \leqslant)$ of a set into a preordered set $(Y, \leqslant)$, we define a mapping $f^{\mathrm{op}} : X \to (Y, \geqslant)$ taking the same values as f; given a mapping $f : (X, \leqslant) \to Y$ of a preordered set $(X, \leqslant)$ into a set Y, we define $f_{\mathrm{op}} : (X, \geqslant) \to Y$ taking the same values; finally, if $f : (X, \leqslant) \to (Y, \leqslant)$ is a mapping between preordered sets, we define $f^{\mathrm{op}}_{\mathrm{op}} = (f^{\mathrm{op}})_{\mathrm{op}} = (f_{\mathrm{op}})^{\mathrm{op}} : (X, \geqslant) \to (Y, \geqslant)$. For brevity we shall also write these mappings as

$$f^{\mathrm{op}} : X \to Y^{\mathrm{op}},\ f_{\mathrm{op}} : X^{\mathrm{op}} \to Y, \quad \text{and } f^{\mathrm{op}}_{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}.$$

We note that $(f^{\mathrm{op}})^{\mathrm{op}} = f$ and $(f_{\mathrm{op}})_{\mathrm{op}} = f$ whenever defined.

Definition 2.4. A *complete lattice* is an ordered set such that any family $(x_j)_{j\in J}$ of elements possesses a smallest majorant and a largest minorant. We denote them by $\bigvee_{j\in J} x_j$ and $\bigwedge_{j\in J} x_j$, respectively.

A complete lattice must possess a smallest element, to be denoted by $\mathbf{0}$, and a largest element, $\mathbf{1}$.

Definition 2.5. If $f : X \to Y$ is a mapping of a set into another, we define its *graph* as the set

$$\operatorname{graph} f = \{(x,y) \in X \times Y; y = f(x)\}.$$

If Y is preordered, we define also its *epigraph* and its *hypograph* as

$$\operatorname{epi} f = \{(x,y) \in X \times Y; f(x) \leqslant y\}, \ \operatorname{hypo} f = \{(x,y) \in X \times Y; y \leqslant f(x)\}.$$

We shall also need the *strict epigraph* and the *strict hypograph*,

$$\operatorname{epi}_s f = \{(x,y) \in X \times Y; f(x) < y\}, \ \operatorname{hypo}_s f = \{(x,y) \in X \times Y; y < f(x)\},$$

of a function $f : X \to Y$, where $a < b$ means that $a \leqslant b$ and $a \neq b$.

Obviously $\operatorname{epi} f = \operatorname{hypo} f^{\mathrm{op}}$.

If X and Y are given, any mapping $X \to Y$ is determined by its graph, and, if Y is an ordered set, also by its epigraph as well as by its hypograph. It is often convenient to express properties of mappings in terms of their epigraphs or hypographs; for examples, see Proposition 4.3 and formulas (5.4).

Definition 2.6. If two preordered sets X and Y and a mapping $f : X \to Y$ are given, we shall say that f is *increasing* if

$$\text{for all } x, x' \in X,\ x \leqslant_X x' \Rightarrow f(x) \leqslant_Y f(x'),$$

and that f is *coincreasing* if

$$\text{for all } x, x' \in X,\ f(x) \leqslant_Y f(x') \Rightarrow x \leqslant_X x'.$$

Finally f is said to be *decreasing* or *codecreasing* if f^{op} (equivalently f_{op}) is increasing or coincreasing, respectively.

If f is increasing, then so is $f^{\mathrm{op}}_{\mathrm{op}}$, whereas f^{op} and f_{op} are decreasing.

The terms *increasing* and *decreasing* are widely used. Birkhoff [3, p. 2], Blyth and Janowitz [6, p. 6], Blyth [5, p. 5], and Grätzer [16, p. 20] call an increasing mapping *order-preserving* or *isotone*. Blyth and Janowitz [6, p. 6] and Blyth [5, p. 5] call a decreasing mapping *order-inverting* or *antitone*. Gierz et al. used *order-preserving* and *monotone* (2003, p. 5) as well as *antitone* (2003, p. 35).

The term *coincreasing* appears in my lecture notes (2002a, p. 12).

To emphasize the symmetry between the two notions, we define, given any mapping $f : X \to Y$ between preordered sets, a preorder $\leqslant_f$ in X by the requirement that $x \leqslant_f x'$ if and only if $f(x) \leqslant_Y f(x')$. Then f is increasing if and only if $\leqslant_X$ is finer than $\leqslant_f$, and f is coincreasing if and only if $\leqslant_X$ is coarser than $\leqslant_f$.

A comparison with topology is in order here. If $f : X \to Y$ is a mapping of a topological space X into a topological space Y with topologies (families of open sets) τ_X and τ_Y, we can define a new topology τ_f in X as the family of all sets $\{x \in X; f(x) \in V\}, V \in \tau_Y$. Then f is continuous if and only if τ_X is finer than τ_f.

Definition 2.7. A mapping $f : L \to M$ of a complete lattice L into a complete lattice M is said to be a *dilation* if $f(\bigvee_{j\in J} x_j) = \bigvee_{j\in J} f(x_j)$ for all families $(x_j)_{j\in J}$ of elements in L.

A mapping is said to be an *erosion* if $f^{\mathrm{op}}_{\mathrm{op}}$ is a dilation, i.e., if $f(\bigwedge_{j\in J} x_j) = \bigwedge_{j\in J} f(x_j)$ for all families $(x_j)_{j\in J}$.

A mapping is said to be an *anti-erosion* if f^{op} is an erosion, i.e., if $f(\bigwedge_{j\in J} x_j) = \bigvee_{j\in J} f(x_j)$ for all families $(x_j)_{j\in J}$.

A mapping is said to be an *anti-dilation* if f^{op} is a dilation, i.e., if $f(\bigvee_{j\in J} x_j) = \bigwedge_{j\in J} f(x_j)$ for all families $(x_j)_{j\in J}$.

We note that a dilation must satisfy $f(\mathbf{0}_L) = \mathbf{0}_M$, an erosion $f(\mathbf{1}_L) = \mathbf{1}_M$.

Matheron in his pioneering treatise (1975:17) used the terms *dilatation* and *erosion* for operations $\mathscr{P}(\mathbf{R}^n) \to \mathscr{P}(\mathbf{R}^n)$. Serra [36,38] defined dilations and erosions as here in the case of complete lattices with $L = M$; anti-erosions and anti-dilations were introduced by Serra [37].

Singer [40, p.172] uses the term *duality* for an anti-erosion. The study of dualities in the sense of Singer is therefore equivalent to that of dilations or erosions.

An explanation for the terms *dilation* and *erosion* is furnished by the operations on subsets of an abelian group G:

$$\delta(A) = A + S, \quad \varepsilon(B) = \{x; x + S \subset B\}, \quad A, B \in \mathscr{P}(G),$$