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A decomposition technique for reconstructing discrete sets from four projections

Péter Balázs *

Department of Computer Algorithms and Artificial Intelligence, University of Szeged, Árpád tér 2, H-6720 Szeged, Hungary

Abstract

The reconstruction of discrete sets from four projections is in general an NP-hard problem. In this paper we study the class of decomposable discrete sets and give an efficient reconstruction algorithm for this class using four projections. It is also shown that an arbitrary discrete set which is Q-convex along the horizontal and vertical directions and consists of several components is decomposable. As a consequence of decomposability we get that in a subclass of *hv*-convex discrete sets the reconstruction from four projections can also be solved in polynomial time. Possible extensions of our method are also discussed.

Keywords: Discrete tomography; Reconstruction algorithm; Decomposable discrete set; Q-convexity; *hv*-Convexity

1. Introduction

One of the most frequently studied problems in discrete tomography is the reconstruction of 2-dimensional discrete sets (the finite subsets of $\mathbb{Z}^2$) from few (usually up to four) projections. Reconstruction algorithms have a wide area of applications (e.g., in electron microscopy, image processing, non-destructive testing, radiology) [1]. Several theoretical questions are also connected with reconstruction such as consistency and uniqueness (as a summary see [2,3]). Since the reconstruction problem is usually underdetermined the number of solutions for a given reconstruction task can be very large. Moreover, the reconstruction under certain circumstances can be NP-hard (see [3,4]). In order to keep the reconstruction process tractable and to reduce the number of solutions a commonly used technique is to suppose having some a priori information of the set to be reconstructed. The most frequently used properties are connectedness, directedness and some kind of discrete versions of the convexity. A lot of work have been done in designing efficient reconstruction algorithms for different classes of discrete sets (e.g., [5–12]). However, up to now only few papers deal with the problem of reconstruction if it is known in advance that the discrete set to be reconstructed consists of several components. While the reconstruction from two projections in the class of *hv*-convex sets is in general NP-hard [4] it turned out that the additional prior knowledge that the set has only one component (i.e., it is 4-connected) leads to a polynomial-time reconstruction algorithm [8]. Later, this algorithm was generalised for the class $\mathcal{S}_8$ of 8-connected *hv*-convex sets, too [5]. Surprisingly, in [13] the authors showed that in this class the prior information that the set to be reconstructed consists of more than one component (i.e., if the set is not 4-connected) leads to a more efficient reconstruction algorithm than the general one developed for the class $\mathcal{S}_8$. These results show the importance of investigating the reconstruction in classes of discrete sets having several components. In this paper we present a class of discrete sets consisting of several components, the class of decomposable discrete sets and study the reconstruction problem in this class.

* Tel.: +36 62 546 396; fax: +36 62 546 397.
E-mail address: pbalazs@inf.u-szeged.hu

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This article is an extended version of the conference paper [14]. The structure of this contribution and the main novelties compared to [14] are the followings. First, the necessary definitions are introduced in Section 2. In Section 3 we define the class of decomposable discrete sets and prove some important properties of sets belonging to this class. While in [14] we had the restriction that a decomposable discrete set cannot have an empty row or column in this paper this restriction is omitted. Moreover, in this contribution we introduce the concept of centre by which a more exhausted study of the decomposable discrete sets is possible. According to this the polynomial-time reconstruction algorithm developed in [14] is redesigned. The results of Sections 4 and 5 are totally new, they are not represented in the conference paper. In Section 4 we show that every discrete set which is Q-convex along the horizontal and vertical directions is also decomposable if it has at least two components. Applying the results of Section 3 we get an O(*mn*) algorithm for the reconstruction problem in this class using four projections. In Section 5 we discuss the possibility to adapt our decomposition technique to *hv*-convex sets to facilitate the reconstruction. Finally, in Section 6 we conclude our results and discuss some possible extensions of our work.

2. Definitions and notation

The finite subsets of $\mathbb{Z}^2$ are called *discrete sets*. Let $\hat{F} = (\hat{f}_{ij})_{m\times n}$ be a binary matrix where $m,n \geqslant 1$ and F denote the set of positions (i,j) where $\hat{f}_{ij} = 1$, i.e., $F = \{(i,j)|\hat{f}_{ij} = 1\}$. Due to the strong relation between binary matrices and discrete sets F is called a discrete set represented by $\hat{F}$, and its elements are called *points* or *positions*. Without loss of generality in the followings we will always assume that there is at least one non-zero element in both the last row and column of $\hat{F}$, i.e., there exist $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ such that $\hat{f}_{i,n} = \hat{f}_{m,j} = 1$. The *kth diagonal/antidiagonal* ($k = 1,\ldots,m+n-1$) of $\hat{F}$ is defined by the set D_k/A_k, respectively, where

$$D_k = \{(i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n\} | i + (n-j) = k\}, \quad (1)$$

$$A_k = \{(i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n\} | i + j = k + 1\}. \quad (2)$$

Let $\mathscr{F}$ denote the class of discrete sets. For any discrete set $F \in \mathscr{F}$ we define the functions $\mathscr{H}, \mathscr{V}, \mathscr{D}$, and $\mathscr{A}$ as follows. $\mathscr{H} : \mathscr{F} \to \mathbb{N}_0^m, \mathscr{H}(F) = H = (h_1,\ldots,h_m)$, where

$$h_i = \sum_{j=1}^{n} \hat{f}_{ij}, \qquad i = 1,\ldots,m, \quad (3)$$

$\mathscr{V} : \mathscr{F} \to \mathbb{N}_0^n, \mathscr{V}(F) = V = (v_1,\ldots,v_n)$, where

$$v_j = \sum_{i=1}^{m} \hat{f}_{ij}, \qquad j = 1,\ldots,n, \quad (4)$$

$\mathscr{D} : \mathscr{F} \to \mathbb{N}_0^{m+n-1}, \mathscr{D}(F) = D = (d_1,\ldots,d_{m+n-1})$, where

$$d_k = \sum_{(i,j)\in D_k} \hat{f}_{ij} = |F \cap D_k|, \qquad k = 1,\ldots,m+n-1, \quad (5)$$

and $\mathscr{A} : \mathscr{F} \to \mathbb{N}_0^{m+n-1}, \quad \mathscr{A}(F) = A = (a_1,\ldots,a_{m+n-1})$, where

$$a_k = \sum_{(i,j)\in A_k} \hat{f}_{ij} = |F \cap A_k|, \qquad k = 1,\ldots,m+n-1. \quad (6)$$

The vectors H, V, D, and A are called the *horizontal*, *vertical*, *diagonal*, and *antidiagonal projections* of F, respectively. The *cumulated vectors* of F are denoted by $\widetilde{H} = (\widetilde{h}_1,\ldots,\widetilde{h}_m)$, $\widetilde{V} = (\widetilde{v}_1,\ldots,\widetilde{v}_n)$, $\widetilde{D} = (\widetilde{d}_1,\ldots,\widetilde{d}_{m+n-1})$, and $\widetilde{A} = (\widetilde{a}_1,\ldots,\widetilde{a}_{m+n-1})$, and defined by the following formulas (see Fig. 1).

$$\widetilde{h}_i = \sum_{l=1}^{i} h_l, \qquad i = 1,\ldots,m, \quad (7)$$

$$\widetilde{v}_j = \sum_{l=1}^{j} v_l, \qquad j = 1,\ldots,n, \quad (8)$$

$$\widetilde{d}_k = \sum_{l=1}^{k} d_l, \qquad \widetilde{a}_k = \sum_{l=1}^{k} a_l, \qquad k = 1,\ldots,m+n-1. \quad (9)$$

Given a class $\mathscr{G} \subseteq \mathscr{F}$ of discrete sets we say that the discrete set $F \in \mathscr{G}$ is *unique in the class* $\mathscr{G}$ (with respect to some projections) if there is no different discrete set $F' \in \mathscr{G}$ with the same projections.

Two points $P = (p_1,p_2)$ and $Q = (q_1,q_2)$ in a discrete set F are said to be *4-adjacent* if $|p_1 - q_1| + |p_2 - q_2| = 1$. The points P and Q are said to be *8-adjacent* if they are 4-adjacent or ($|p_1 - q_1| = 1$ and $|p_2 - q_2| = 1$). The sequence of distinct points $P_0,\ldots,P_k$ is a *4/8-path* from point P_0 to point P_k in a discrete set F if each point of the sequence is in F and P_l is 4/8-adjacent, respectively, to P_{l-1} for each $l = 1,\ldots,k$. Two points are 4/8-connected in the discrete set F if there is a 4/8-path, respectively, in F between them. A discrete set F is *4/8-connected* if any two points in F are 4/8-connected, respectively, in F. The 4-connected set is also called as polyomino [15]. The discrete set F is *horizontally convex/vertically convex* (or shortly, *h*-convex/*v*-convex) if its rows/columns are 4-connected, respectively. The *h*- and *v*-convex sets are called *hv*-convex (see Fig. 1). In this paper we are going to study the reconstruction problem

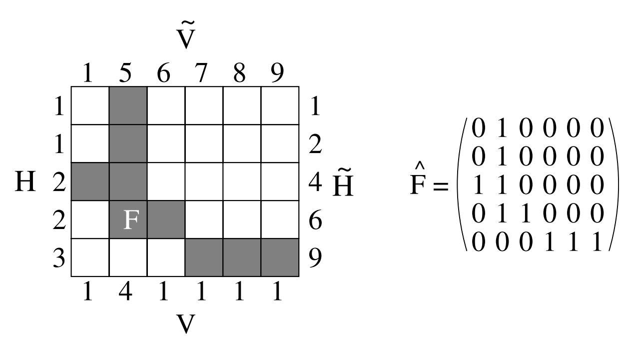

Fig. 1. An *hv*-convex 8- but not 4-connected discrete set F and the corresponding binary matrix $\hat{F}$. The elements of F are marked with grey squares. The projections of F are H, V, $D = (0,0,0,0,2,2,3,2,0,0)$, and $A = (0,1,2,1,1,1,0,1,1,1)$. The cumulated vectors of F are $\widetilde{H}, \widetilde{V}, \widetilde{D} = (0,0,0,0,2,4,7,9,9,9)$, and $\tilde{A} = (0,1,3,4,5,6,6,7,8,9)$.