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Algorithms for the computation of the Minkowski functionals of deterministic and random polyconvex sets

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Abstract

We give algorithms for the simultaneous computation of the area, boundary length and connectivity (the so-called *Minkowski functionals*) of binary images. It is assumed that a binary image is a discretization of a two-dimensional *polyconvex* set which is a union of convex components. Edge-corrected versions of these algorithms are used for the estimation of specific intrinsic volumes of a stationary random closed set from a single realization given by a binary image. Performance and exactness of the algorithms in two dimensions are discussed on numerical examples. Comparison to other known methods is provided.

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1. Introduction

Minkowski functionals such as the volume, the surface area and the Euler–Poincaré characteristic are set functions that describe the geometric and topological structure of a *regular* set pretty well; cf. e.g. [1]. Known in integral and convex geometry also as *intrinsic volumes* or *quermaß integrals* and in physics and differential geometry as *curvature measures*, these functionals are used in morphological image analysis for the characterization of binary and gray-scale images. They are successfully applied to various problems in astronomy, materials science, medicine, and biology; cf. [2–8].

Beginning with the pioneering work of Serra [9], several approaches to the computation of single Minkowski functionals such as the Euler number have been proposed e.g. in [5,10,11]. In recent papers [12–14], alternative methods are proposed allowing for the simultaneous computation of all Minkowski functionals of sets from different regularity classes such as *polyconvex sets* or *sets with positive reach*. The above methods can be modified to estimate the specific intrinsic volumes of stationary random closed sets by introducing an appropriate edge correction. In this case, it is assumed that the given data is part of a realization of an unbounded spatially homogeneous random closed set observed within a bounded sampling window. Alongside with the development of theoretical methods, the algorithmic issues of the computation of Minkowski functionals of discrete sets given on lattices or *binary images* have been touched upon in [5,15–19].

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In the present paper, we embed the methods proposed in [12,13] in the general context provided by the theorem of Hadwiger and the method of moments; see Sections 2.2 and 3.2 for details. Furthermore, we discuss the corresponding algorithms for these methods in Section 2.4 and give their edge-corrected counterparts for the estimation of specific intrinsic volumes of random sets in Section 3.3. A comparison of the numerical results with those of methods from [5,19] is provided in Sections 2.5 and 3.5. The paper concludes with a brief discussion of numerical results in Section 4.

In the theoretical part of the paper (Sections 2.1 and 2.2, 3.1 and 3.2), binary images of arbitrary dimension $d \geqslant 2$ are considered. In this way, we emphasize that the proposed computational methods are independent of d. However, the implementation clearly depends on the dimension. This is the reason why we give the algorithms and numerical results in dimension $d = 2$ only. The implementation of the algorithms in three dimensions and the corresponding numerical studies are ongoing research. They will be considered in a forthcoming paper.

The proposed algorithms seem to be relatively complex in comparison with other known methods. As a consequence, they are slower than e.g. the very efficient algorithms given in [9]. This is the price one has to pay for the nice statistical properties of these methods that allow their use in image comparison based on asymptotical Gauss tests (see [20] for details).

2. Computation of intrinsic volumes for deterministic sets

In this section, we discuss methods for the computation of Minkowski functionals for deterministic sets. For the sake of convenience, we rather use *intrinsic volumes* which differ from Minkowski functionals by a constant factor and by the inverse order of notation; cf. [21]. First, preliminaries on intrinsic volumes are given. Then, computational methods based on Hadwiger's expansion are described for arbitrary dimensions d. In Section 2.3, binary images are introduced as discretizations of polyconvex sets in $\mathbb{R}^2$. An algorithm for the computation of intrinsic volumes of binary images is given in Section 2.4. Finally, numerical results are discussed.

2.1. Intrinsic volumes of polyconvex sets

Let $\mathcal{K}$ be the set of all compact convex subsets of $\mathbb{R}^d$. A set is called *polyconvex* if it is a finite union of sets from $\mathcal{K}$. The class $\mathcal{R}$ of all polyconvex sets in $\mathbb{R}^d$ is often called the *convex ring*. This set family is general enough to model most objects in image analysis in the sense of approximation of these objects by unions of polyhedra as explained in Section 2.3 for the two-dimensional case. Let $A \oplus B = \{x + y\colon x \in A, y \in B\}$ be the *Minkowski sum* of sets A and B. Denote by $B_r(x)$ the ball of radius $r > 0$ centered in $x \in \mathbb{R}^d$. For any set $K \subset \mathbb{R}^d$, the set of inner points of K in $\mathbb{R}^d$ is denoted by $\mathrm{int}(K) = K \setminus \partial K$, where ∂K is the boundary of K.

For a set $K \in \mathcal{K}$, the *intrinsic volumes* $V_0(K), \ldots, V_d(K)$ are usually introduced as coefficients in the polynomial expansion of the volume (or Lebesgue measure) $V_d(K \oplus B_r(o))$ of the *parallel neighborhood* $K \oplus B_r(o)$ of K with respect to r

$$V_d(K \oplus B_r(o)) = \sum_{j=0}^{d} r^{d-j} \kappa_{d-j} V_j(K), \tag{1}$$

where $\kappa_j = V_j(B_1(o))$ is the j-volume of the j-dimensional unit ball. By additivity, V_j can be extended in a unique way to $\mathcal{R}$, namely,

$$V_j(K_1 \cup \cdots \cup K_n) = \sum_{i=1}^{n} (-1)^{i-1} \sum_{1 \leqslant j_1 < \cdots < j_i \leqslant n} V_j(K_{j_1} \cap \cdots \cap K_{j_i}) \tag{2}$$

for any $K_1, \ldots, K_n \in \mathcal{K}$ and $j = 0, \ldots, d$. It can be shown that for any $K \in \mathcal{R}$ $2V_{d-1}(K)$ is the surface area and $V_0(K)$ is the Euler–Poincaré characteristic, that is, a linear combination of *Betti numbers* of K; see e.g. [1,11]. In two dimensions, $V_0(K)$ is equal to the number of "clumps" minus the number of "holes" in $K \in \mathcal{R}$. For compact sets K with C^2-smooth boundary ∂K, intrinsic volumes $V_j(K)$ are integrals of mean curvature of ∂K, cf. [1].

Formula (1) is often referred to as the *Steiner formula*. It is a special case of the well-known result of Hadwiger (cf. [22]): any additive rigid motion invariant continuous functional F on $\mathcal{R}$ can be represented as a linear combination of intrinsic volumes, i.e.,

$$F(K) = \sum_{j=0}^{d} a_j V_j(K), \quad K \in \mathcal{R} \tag{3}$$

with coefficients $a_0, \ldots, a_d \in \mathbb{R}$. These coefficients can be defined from the following system of linear equations

$$F(K_i) = \sum_{j=0}^{d} a_j V_j(K_i), \quad i = 0, \ldots, d, \tag{4}$$

where $K_0, \ldots, K_d$ are simple convex bodies (for instance, a ball, a segment, a square, a cube, and so on) for which the values of F and of V_j can easily be computed and the matrix $(V_j(K_i))$ in (4) is not singular.

2.2. Computation of intrinsic volumes

Suppose that there exist functionals $F_0, \ldots, F_n$, $n \geqslant d$ on $\mathcal{R}$ satisfying the assumptions of the characterization theorem of Hadwiger with coefficients a_{ij} from the expansion

$$F_i(K) = \sum_{j=0}^{d} a_{ij} V_j(K), \quad i = 0, \ldots, n, \quad K \in \mathcal{R} \tag{5}$$

that are either known or can be assessed by method (4). If $n = d$ and the matrix $A = (a_{ij})$ is regular then the vector $V(K) = (V_0(K), \ldots, V_d(K))^\top$ of intrinsic volumes of any polyconvex set K is the solution of the system of equations (5). In the matrix form, it holds $V(K) = A^{-1} F(K)$, where