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Modeling multi-criteria objective functions using fuzzy measures

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ABSTRACT

We introduce the idea of a measure. We describe several important finite integrals useful for obtaining an average value of a collection of argument values weighted by a measure. We particularly look at the case of binary measures and show that all integrals in this case evaluate to the same value. We describe the use of measures in multi-criteria decision making as a way of expressing a decision maker's objective function in terms of collection of relevant criteria. We look at the role of an integral as a way to evaluate an alternative's overall satisfaction to their objective function in terms of its satisfaction to the individual criteria. We look at a number of special types of measure based decision objective functions.

1. Introduction

Multi-criteria decision problems are pervasive. They arise in such common tasks as deciding what journal you want to submit a paper to and such complex tasks as a drone tracking a target. In these applications the decision maker has some objective related to a relevant set of multiple criteria. In many cases the decision maker's objective function can be expressed in terms of a fuzzy measure μ over the set of multiple criteria. Here we focus on the use of a measure to represent these decision maker objective functions. For example in deciding which journal to submit his paper a researcher may consider the following criteria relevant high impact factor, fast review time and good chance for acceptance. One formulation for his objective function may be to select a journal that satisfies all these criteria. Another formulation for his objective could be to associate with each criterion an importance weight and then look for the alternative that provides maximal weighted satisfaction to his criteria. As will be clear either of these objectives can be expressed in terms of a fuzzy measure over the set of criteria.

In addition to their objective function a decision maker has available a collection of possible alternative actions that they can take. Here we must decide which alternative action to choose, based upon its satisfaction to the decision maker's objective function as manifested through an alternatives satisfaction to the individual criteria. We describe how we can obtain the satisfaction of an alternative to the decision maker's objective function by calculating the integral, with respect to the measure μ, of the alternatives satisfactions to relevant criteria. Here the integral provides a kind weighted average of individual criteria satisfactions were the weights are determined by the fuzzy measure modeling the decision maker's objective function.

A number of researchers have worked on issues related to fuzzy measures in multi-criteria decision making among these are the following; Greco [1–4], Grabisch [5–7], Bustince [8,9], Torra [10,11] and Marichal [12].

The paper is organized as follows. We introduce the concept of a fuzzy measure and describe some of its properties. We provide some integrals that can be used for aggregating a collection of argument values weighted by a measure. We then discuss how fuzzy measures can be used in multi-criteria decision making to model a decision maker's objective function in terms of collection criteria. We then look at the role of some notable measures in context multi-criteria decision-making.

2. Fuzzy measures and their integrals

A fuzzy measure [13–16] on a finite space $X = \{x_1, \ldots, x_n\}$ is a mapping $\mu: 2^X \to [0, 1]$ having the properties:

1. $\mu(\varnothing) = 0$, **2**. $\mu(X) = 1$ and **3**. $\mu(A) \geqslant \mu(B)$ if $A \supset B$

Thus a measure associates with subsets of X a value from the unit interval that is monotonic in the sense that a smaller set cannot have a bigger value than a larger set. In the following we shall follow the policy of simply using the term measure with the understanding that we are referring to a fuzzy measure.

The prototypical example of measure is the additive measure. Here each element x_j has an associated value $\alpha_j \geqslant 0$ where $\sum_{j=1}^{n} \alpha_j = 1$. For this additive measure for any subset A, $\mu(A) = \sum_{x_j \in A} \alpha_j$. One special case of additive measure is one where

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$\alpha_j = 1/n$ for all x_j. Another special case of additive measure is one in which $\alpha_k = 1$ and $\alpha_j = 0$ for all $j \neq k$.

Assume μ_1 and μ_2 are two measures on the space X if $\mu_1(A) \geqslant \mu_2(A)$ for all A we say $\mu_1 \geqslant \mu_2$, μ_1 is bigger than μ_2. At times, in this work, we shall find it convenient to say that μ_1 is more optimistic than μ_2 if $\mu_1 \geqslant \mu_2$.

If μ is a measure on X we define its dual as a measure $\hat{\mu} : X \to [0, 1]$ defined so that $\hat{\mu}(A) = 1 - \mu(\bar{A})$. We note duals come in pairs in that $\hat{\hat{\mu}}(A) = \mu(A)$.

The two special measures are μ^* defined such that $\mu^*(\varnothing) = 0$ and $\mu^*(A) = 1$ for $A \neq \varnothing$ and the measure μ_* defined such that $\mu_*(X) = 1$ and $\mu_*(A) = 0$ for $A \neq X$. These are respectively the biggest and smallest measures, that is for any measure μ we have $\mu_* \leq \mu \leq \mu^*$. We also note that μ_* and μ^* are duals.

Another interesting special class of measures are the cardinality-based measures. For these measures the value of $\mu(A)$ just depends on the number of elements in μ, independent of which elements are in A. More formally we can define a cardinality-based measure in terms of a collection of parameters, $0 = \beta_0 \leq \beta_1 \leq \ldots \beta_{n-1} \leq \beta_n = 1$ so that $\mu(A) = \beta_{|A|}$ where $|A|$ is the cardinality of A.

We observe that μ_* and μ^* are cardinality-based measures. Also the additive measure where $\alpha_j = 1/n$ for all x_j is also a cardinality-based measure.

In [17] Yager investigated a number of operations can be used to obtain measures from other measures. Assume μ_1 is a measure on X and F: $[0, 1] \to [0, 1]$ is such that.

(1) $F(0) = 0$
(2) $F(1) = 1$
(3) $F(a) \geqslant F(b)$ for $a > b$

then μ defined such that $\mu(A) = F(\mu_1(A))$ is a measure.

An important special case of this is one in which μ_1 is an additive measure. Torra referred to this as deformed additive measures [14].

Assume for $j = 1$ to q that μ_j are a collection of measures on X and G is an aggregation operator. Then μ defined such for all $A \subseteq X$

$$\mu(A) = G(\mu(A), \ldots, \mu_q(A))$$

is a measure. We recall an aggregation operator G is a function of $q > 1$ arguments G: $[0, 1]^q \to [0, 1]$ having the properties [18]

$$G(0, 0, \ldots, 0) = 0$$
$$G(1, 1, \ldots, 1) = 1$$
$$G(a_1, \ldots, a_q) \geqslant G(b_1, \ldots, b_q) \text{ if } a_j \geqslant b_j \text{ for all } j$$

A prototypical example of an aggregation operator is in the weighted average, $G(a_1, \ldots, a_q) = \sum_{j=1}^{q} w_j a_j$ where $w_j \in [0, 1]$ and $\sum_{j=1}^{q} w_j = 1$. There are many other examples of aggregation operators, among the most notable are t-norms, t-conorms and OWA operators [18,19].

Theorem. *Assume μ_i are a collection of q additive measures on X such that $\mu_i(\{x_j\}) = \lambda_{ij}$. Let μ be a measure defined as $\mu(A) = G(\mu_1(A), \ldots, \mu_q(A)) = \sum_{i=1}^{q} w_i \mu_i(A)$ where $w_i \in [0, 1]$ and sum to one. Then μ is also an additive measure with $\mu(\{x_j\}) = \sum_{i=1}^{q} w_i \lambda_{ij}$. Furthermore if A be any subset of X then*

$$\mu(A) = \sum_{i=1}^{q} w_i \mu_i(A) = \sum_{i=1}^{q} \left(w_i \sum_{j \in A} \lambda_{ij} \right) = \sum_{j \in A} \sum_{i=1}^{q} w_i \lambda_j = \sum_{j \in A} \mu(\{x_j\}).$$

Theorem. *Assume μ_i are a collection of q cardinality-based measures on X such that β_{ik} is the measure of μ_i for a set of cardinality k. Assume w_i is a collection of q weights such that each $w_i \in [0, 1]$ and their sum is one. If μ be a measure defined such that for any subset A*

$$\mu(A) = G(\mu_1(A), \ldots, \mu_q(A)) = \sum_{i=1}^{q} w_i \mu_i(A))$$

then $\mu(A)$ is a cardinality-based measure with parameters $\beta_k = \sum_{i=1}^{q} w_i \beta_{ik}$.

Proof. For any A we have $\mu(A) = \sum_{i=1}^{q} w_i \mu_i(A) = \sum_{i=1}^{q} w_i \beta_{i|A|}$.

Let f be a function so that f: $X \to [0, 1]$, it associates with each element in X a value in the unit interval. An integral provides a way for obtaining a weighted average of values of $f(x_i)$ with respect to a weighting of the x_j determined by a measure μ. We shall generically denote this as $\text{Int}_\mu(f)$. One property we desire of these integrals, is monotonicity, if f_1 and f_2 are such that $f_1(x_i) \geq f_2(x_i)$ for all x_i then we desire that $\text{Int}_\mu(f_1) \geq \text{Int}_\mu(f_2)$. We also want these integrals to be mean like [20] with respect to value of f. In particular we want boundedness, $\text{Min}_j[f(x_j)] \leq \text{Int}_\mu(f) \leq \text{Max}_j[f(x_j)]$. An implication of this boundedness is idempotency, of all $f(x_j) = a$ then $\text{Int}_\mu(f) = a$. The integral of f with respect to μ, $\text{Int}_\mu(f)$ is a mean type operation and can be seen as a representative value of f. In [21] Klement, Mesiar, Spizzichino and Stupnanová provide a whole class of these integrals. Wang, Yang and Leung [22] provide a comprehensive discussion of these integrals.

One common form of integral of f with respect to the measure μ is the Choquet integral, $\text{Choq}_\mu(f)$ [22,23]. In describing this and other integrals we shall find it convenient use an index function π so that $\pi(j)$ is index of the value of x_i having the jth largest value of $f(x_i)$. Here then $f(x_{\pi(j)})$ is jth largest value of $f(x)$. Using this index function we have

$$\text{Choq}_\mu(f) = \sum_{j=1}^{n} (\mu(H_j) - \mu(H_{j-1}))\, f(x_{\pi(j)})$$

where $H_j = \{x_{\pi(k)} / k = 1 \text{ to } j\}$, it is the subset of elements in X having the j largest values of f. One interesting feature of this integral is that it essentially provides a simple weighted average of the $f(x_{\pi(j)})$. We see this easily if we denote $w_j = \mu(H_j) - \mu(H_{j-1})$ and since $\mu(H_j) - \mu(H_{j-1}) \geq 0$ these are non-negative and since $\sum_{j=1}^{n} w_j = \mu(X) - \mu(\varnothing) = 1$. We also note that with a little algebra and using the convention that $f(x_{\pi(n+1)} = 0$ we can show that

$$\text{Choq}_\mu(f) = \sum_{j=1}^{n} (f(x_{\pi(j)}) - f(x_{\pi(j+1)}))\mu(H_j)$$

Another notable form of integral is the Sugeno integral $\text{Sug}_\mu(f)$ [13,24]. In this case

$$\text{Sug}_\mu(f) = \operatorname*{Max}_{j=1}^{n}[f(x_{\pi(j)}) \wedge \mu(H_j)].$$

Another form of integral is the Shilkret integral, $\text{Sh}_\mu(f)$ [25]. In this case

$$\text{Sh}_\mu(f) = \operatorname*{Max}_{j=1}^{n}[f(x_{\Pi(j)}) \cdot \mu(H_j)]$$

We note that for any f and μ, $\text{Sug}_\mu(f) \geq \text{Sh}_\mu(f)$.

Another related approach involves the use of the median $\text{Med}_\mu(f)$. In this approach we determine the element $x_{\pi(j)}$ so that $\mu(H_j) \geqslant 0.5 > \mu(H_j)$ and then $\text{Med}_\mu(f) = x_{\pi(j)}$. Actually we can express the median in a form similar to the Sugeno and Shilkret integral. Let M be a function M: $[0, 1] \to [0, 1]$ so that $M(a) = 0$ for $a < 0.5$ and $M(a) = 1$ for $a \geqslant 0.5$. Using this we see that $\text{Med}_\mu(f) = \operatorname*{Max}_{j=1}^{n}[f(x_{\pi(j)}) \wedge M(\mu(H_j))]$. We also can express it as $\text{Med}_\mu(f) = \operatorname*{Max}_{j=1}^{n}[f(x_{\pi(j)}) M(\mu(H_j))]$. Actually more generally we express it as