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Fuzzy m-ary adjacency relations in social network analysis: Optimization and consensus evaluation

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ABSTRACT

The main contribution of this paper consists in extending the 'soft' consensus paradigm of fuzzy group decision making developed under the framework of numerical fuzzy preferences. We address the problem of consensus evaluation by endogenously computing the importance of the decision makers in terms of their influence strength in the network. To this aim, we start from centrality measure and combine it with the fuzzy m-ary adjacency relation approach. In this way, we introduce a flexible consensus measure that takes into account the influence strength of the decision makers according to their eigenvector centrality. Moreover, we propose an optimization problem which determines the maximum number of the most important decision makers that share a fixed desirable consensus level.

1. Introduction

In recent years, social network analysis (SNA) has attracted the interest of scholars in the field of decision making [1] since the network model can be effectively used for modeling interactions between decision makers. Opinion similarity, for example, can be represented by means of an adjacency matrix and several properties and indices developed for this kind of matrices can be translated into the decision making framework. The adjacency matrix is a notion that has a strong interest in SNA, since it represents important relationships between the nodes of the network. Unfortunately, it has two main limitations: the first one is that it can be used only for representing pairwise adjacencies, the second one is that its crisp definition is not suitable to cope with the vagueness that makes adjacency between two nodes be a matter of degree. While the extension of binary adjacency relations to m-ary ones is a problem that did not catch so far the attention of researchers in SNA, some extensions of SNA have been proposed to a formal context in which the network structure can be represented by fuzzy graphs [2], through the fuzzy generalization of the definition of relation [3]. Accordingly, it is possible to take into account at the same time the vagueness influencing the relationships between the actors involved in the social dynamics and the qualitative nature of the variables/attributes. Some other approaches to SNA that use fuzzy relations have been recently proposed. In [4] the notion of regular similarity was represented by a fuzzy binary relation. In [5], human-focused concepts associated with social networks are formalized using set-based relational network theory and fuzzy sets. In [6] a model was proposed for evaluating reciprocity of networks represented by means of fuzzy binary relations.

This paper aims to extend the preliminary results presented in [7,8], where the imprecision permeating the relationships between the nodes of a social network was modeled using fuzzy binary adjacency relations [9] and higher dimensional fuzzy m-ary adjacency relations were constructed from the binary relations by means of OWA functions [10]. This allowed to overcome both limitations previously highlighted and to characterize the attitude of the actors to connect each other moving continuously from non-compensatory to full-compensatory situations. Considering that in [7,8] we assumed that fuzzy binary adjacency relations were given, now we extend our previous approach by assuming that adjacency relations between decision makers are derived from their preferences on a set of alternatives, so that adjacency relations explicitly represent the grade of agreement between the decision makers. The flexible method we propose for deriving the adjacency relations allows us to model different decision problems. In many decision models, different weights are associated to the various decision makers, which can be interpreted, according to the context, as decision power, percentage representation, influence capability, and so forth. Since we assume that decision makers

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are represented by nodes in a social network, a reasonable way to weigh them endogenously is to use a centrality measure. Centrality measure is a core concept in SNA that is used to describe a wide spectrum of theoretical issues ranging from importance to influence to leadership [11–13]. Since adjacency and centrality are two very closely connected notions and both influence the behavior of the actors of social networks when addressing decision making tasks, we consider that a suitable centrality measure is a proper tool to evaluate the relevance of each decision maker in terms of its influence power. More precisely, we choose the eigenvector centrality [14], in analogy to Google's PageRank algorithm, since this centrality measure is particularly suitable to quantify the influence of the network nodes and therefore to be used in a decision making context. The quantification of the influence power of each decision maker allows us to introduce a new approach to consensus evaluation in the set of decision makers. First we order the decision makers according to their influence power, then we compute the internal agreement for all the coalitions of the m most influencing decision makers, according to the paradigm introduced in [15] and using the fuzzy m-ary adjacency relations. By considering all the possible values of m, from 1 to the total number n of decision makers, we obtain a complete evaluation of the consensus in the most influencing subgroups. The results can be reported in a bar chart which provides a simple and effective synthesis of this type of consensus evaluation. Our approach is in the spirit of [15–17], where the so called 'soft' consensus paradigm in fuzzy group decision making has been developed under the standard framework of numerical fuzzy preferences. With soft consensus we basically mean the treatment of both consensus and its formation as gradual notions. The interested reader may also find in [18,19] complete and critical overviews of the different approaches to consensus in fuzzy group decision making, future trends included.

This paper is outlined as follows. Section 2 offers a presentation of valued and fuzzy adjacency relations and shows how it is possible to derive an adjacency relation on a set of decision makers given their preferences on a set of alternatives. In Section 3 the construction of fuzzy m-ary adjacency relations is described, as previously introduced in [7]. In Section 4, after a brief review of the optimization problems already introduced in [7,8], we prove a new result for one of them and propose another optimization problem based on fuzzy m-ary adjacency relations. In the same section we develop our novel approach to consensus evaluation, focused on the most influencing decision makers as characterized by eigenvector centrality. Section 5 presents a commented numerical example on fraud classification in order to show how the proposed optimization problems can be applied and the consensus evaluation can be performed. Finally, Section 6 contains some conclusions as well as our views on possible future research.

2. SNA and adjacency matrix

SNA can be seen as the set of tools and techniques employed to describe and analyze relations between entities [20]. Such entities may be people, organizations, symbols in texts, elements of a data set, geographical regions and so on. The primary focus is on the relations between entities and not on the entities themselves or their attribute description. From the mathematical perspective it is safe to say that the techniques employed in SNA mainly stem from graph theory and linear algebra. In fact, in SNA, the main tool to represent the relationships between social objects is the adjacency matrix. Hereafter, we will consider a set $D = \{d_1, \ldots, d_n\}$ of decision makers (DMs). Then, an adjacency matrix $\mathbf{A} = (a_{ij})_{n\times n}$ with $a_{ij} := \mu_A(d_i, d_j)$ is a representation of an adjacency relation, $A \subseteq D \times D$, whose characteristic function is $\mu_A : D \times D \to \{0,1\}$ such that

$$\mu_A(d_i, d_j) = \begin{cases} 1, & \text{if } d_i \text{ is related to } d_j, \\ 0, & \text{if } d_i \text{ is not related to } d_j. \end{cases}$$

By definition [9], adjacency relations satisfy properties of reflexivity, $\mu_A(d_i, d_i) = 1\ \forall i$, and symmetry, $\mu_A(d_i, d_j) = \mu_A(d_j, d_i)\ \forall i, j$. Note that, unlike in equivalence relations, no transitivity condition is required to hold.

One of the main characteristics of matrix $\mathbf{A}$ is that it is a concise synthesis of the pairwise relationships between elements in D, but due to the fact that it does not take into account the strength of the relationships, it could happen that, by using such type of adjacency matrix, very different cases are treated in the same way, without discriminating among situations where intensities of relationship may be very different. This can seriously weaken the analysis of a social network.

2.1. Valued and fuzzy adjacency relations

A fuzzy binary relation on a single set, hereafter called fuzzy relation if not differently stated, is a fuzzy subset of the Cartesian product, i.e. a relation $R_2 \subseteq D \times D$ defined through the following membership function

$$\mu_{R_2} : D \times D \to [0, 1]. \tag{1}$$

Here also, by putting $r_{ij} := \mu_{R_2}(d_i, d_j)$, a fuzzy relation can be suitably represented by a matrix $\mathbf{R} = (r_{ij})_{n\times n}$ where the value of each entry is the degree to which the relation between d_i and d_j holds. In other words, the value of $\mu_{R_2}(d_i, d_j)$ is the degree of truth of the statement: 'd_i and d_j are related'. Therefore, in the context of SNA

$$\mu_{R_2}(d_i, d_j) = \begin{cases} 1, & \text{if } d_i \text{ has the strongest possible} \\ & \text{degree of relationship with } d_j, \\ \gamma \in]0,1[& \text{if } d_i \text{ is, to some extent, related with } d_j, \\ 0, & \text{if } d_i \text{ is not related with } d_j. \end{cases}$$

Moreover, let us remark that, in literature, the term adjacency relation has often been considered interchangeable with tolerance [21], proximity [18], and compatibility [9].

Fuzzy adjacency relations, as well as crisp adjacency relations, are reflexive and symmetric. It is useful to spend some words about symmetry. The assumption of symmetry, i.e. $\mu_{R_2}(d_i, d_j) = \mu_{R_2}(d_j, d_i)\ i, j = 1, \ldots, n$, is of great help for the model because it allows such relations to be represented by means of undirected graphs. Furthermore, in many real-world cases, symmetry is spontaneously satisfied by the nature of the relationship. In fact, it will be clear that a relation of consensus genuinely satisfies the condition of symmetry.

As for the case of crisp adjacency relations, fuzzy adjacency relations are not necessarily transitive. For this reason we remark the difference between them and similarity relations, i.e. similarity relations are fuzzy equivalence relations, which, conversely, are defined to be transitive [3].

According to its definition, a fuzzy adjacency relation seems to be a special case of a valued adjacency relation. That is, the set [0,1] is not a necessary condition although is usually included in the definition. As it has been remarked in literature [22], the unit interval can consistently be substituted by any lattice, L, and therefore, the membership function can be generalized,

$$\mu_{R_2} : D \times D \to L. \tag{2}$$

Thus, valued adjacency relations are easily meant to be fuzzy adjacency relations in their broader sense. In spite of this remark, whose aim was that of underlying (i) the non-restrictive nature of the range [0,1] and (ii) the extendibility of the results contained in this paper, for sake of simplicity, we are going to use the real unit interval.