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A generalized framework for mean aggregation: Toward the modeling of cognitive aspects

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ABSTRACT

We provide an overview of mean/averaging operators. We introduce the basic OWA operator and look at some cases of the generalized OWA operator. We next look at the issue of importance weighted mean aggregation. We provide a generalized formulation using a fuzzy measure to convey information about the importances of the different arguments in the aggregation. We look at some different measures and the associated importance formulation they manifest. We further generalize our formulation by allowing for the inclusion of an attitudinal aggregation function. This allows us to implement many different types of aggregation including Max, Min and Median. Finally we provide a simple parameterized formulation for generalized class of mean operators.

1. Introduction

Consensus, which can be seen as a type of information fusion, involves a process of getting a group of agents to agree upon a solution to some problem [1–5]. Often in these types of problems each of the individual agents have their own suggested solution value. One important task in the formulation of a consensus is the construction of a value or solution as a proposed answer [5]. This task often involves an aggregation [6–8] of the different proposals provided by the individual participants. An important property of any process used in this type of aggregation is idempotency; if all the participating suggest the same solution this should clearly be a good consensus solution. Aggregation operators that manifest this idempotency are referred to as mean or averaging operators. In support of this task we suggest a generalized framework for implementing the mean operator and look at various formulations that are special cases of this framework. Our focus here to eventually enable the mathematical modeling of linguistically expressed mean type aggregation imperatives. With this in mind we try, as much as possible, to relate choice of parameters to cognitive concepts.

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2. Overview of mean operators

Assume v_j are a collection of n numeric values. An important type of aggregation that can be performed on these values is an averaging. An average or mean, here we shall use the terms average and mean synonymously, provides a kind of representative value for the collection. The prototypical example of a mean aggregation is the arithmetic mean, $\bar{v} = \frac{1}{n}\sum_{j=1}^{n} v_j$. However this is not the only mean type operation, Bullen [9] has a monograph on mean aggregation covering numerous types of means. In [6–9] the authors provide a useful discussion of mean aggregation.

A formal definition of a mean aggregation operator is the following.

Definition. A mapping $F{:}R^n \rightarrow R$ is called a mean operator if it has the following properties:

(1) **Symmetry**: $F(v_1, \ldots, v_n) = F(v_{\pi(1)}, \ldots, v_{\pi(n)})$, where π is a permutation.
(2) **Monotonicity**: $F(v_1, \ldots, v_n) \geqslant F(b_1, \ldots, b_n)$ if $v_j \geqslant b_j$ for all j.
(3) **Boundedness**: $\text{Min}_j[v_j] \leqslant F(v_1, \ldots, v_n) \leqslant \text{Max}_j[v_j]$.

The boundedness is the key defining property of the mean operator as many types of aggregation operators have the first two properties.

An immediate implication of the boundedness condition is the idempotency of the mean, if all $v_i = v$ then $F(v_1, \ldots, v_n) = v$. We note that in some cases the mean is defined using idempotency instead of boundedness. As we easily see in the following idempotency and monotonicity imply boundaries. Consider $F(v_1, \ldots, v_n)$, from monotonicity $F(v_1, \ldots, v_n) \leqslant F(v^*, \ldots, v^*)$, where $v^* = \text{Max}_j[v_j]$ and from idempotency $F(v^*, \ldots, v^*) = v^*$. Similarly with $v_* = Min_j[v_j]$ then $F(v_1, \ldots, v_n) \geqslant F(v_*, \ldots, v_*) = v^* = \text{Min}[v_j]$.

We also note that from symmetry and monotonicity we can get a more relaxed version of monotonicity. Assume for $j = 1$ to n we have v_j and b_j. Let π be a permutation of the indices such that $v_{\pi(j)} \geqslant b_j$ for all j then $f(v_{\pi(1)}, \ldots, v_{\pi(n)}) \geqslant f(b_1, \ldots, b_n)$. However from symmetry $f(v_1, \ldots, v_n) = f(v_{\pi(1)}, \ldots, v_{\pi(n)}) \geqslant f(b_1, \ldots, b_n)$. Thus if we have two sets of n values and if we can index them so that $v_j \geqslant b_j$ then the aggregation of the v_j's is at least as big as the aggregation of the b_j's.

Example. Assume we grade a person by taking a mean of three tests T_1, T_2, T_2, Grade = $F(T_1, T_2, T_3)$. Consider two people x and y and let us denote their scores on the test as $T_j(x)$ and $T_j(y)$ for j = 1–3. Assume the scores are as follows:

$$\begin{array}{ll} T_1(x) = 5 & T_1(y) = 6 \\ T_2(x) = 8 & T_2(y) = 2 \\ T_3(x) = 7 & T_3(y) = 3 \end{array}$$

If we permute the scores of y we can get

$$\begin{array}{ll} T_1(x) = 5 & T_2(y) = 2 \\ T_2(x) = 8 & T_1(y) = 6 \\ T_3(x) = 7 & T_3(y) = 3 \end{array}$$

We see whatever mean we use Grade(x) $\geqslant$ Grade(y).

One important property not generally available in mean operators is associativity. We recall a binary argument function f is associative if $f(f(v_1, v_2), v_3) = f(v_1, f(v_2, v_3)) = f(v_1, v_2, v_3)$. One benefit of associativity is that we can start with the definition of the aggregation of two arguments and the definition of the aggregation of any number of arguments is determined. A second feature, one more useful for calculations, is that if we have the aggregation for n arguments we can use this directly to obtain the aggregation of $n + 1$ arguments

$$f(v_1, \ldots, v_n, v_{n+1}) = f(f(v_1, \ldots, v_n), v_{n+1}))$$

However, it is the loss of the first feature that is of more concern when defining averaging operators especially in situations in which we use the operators in aggregations with different cardinality of arguments, here we must provide a definition of the aggregation for each of the possible cardinalities. That is we must describe the aggregation process for each value of n. While we can arbitrarily assign a different type of mean operator for each n we normally desire some "consistency" between the aggregations at different argument cardinalities. The classical mean, $\bar{v} = \frac{1}{n}\sum_{j=1}^{n} v_j$, illustrates this. Here we have expressed how to find the average for each n and have done it is what appears a consistent manner.

In the literature there are many forms of mean aggregation operations, many of which are defined in a consistent matter over different cardinalities of arguments [9]. While many of these are available over the whole range of argument values others may require non-negative arguments. In the following we shall assume the arguments are all non-negative. Among some notable mean operators, in addition to the basic arithmetic average, are the following:

$$F(v_1, \ldots, v_n) = n\left(\sum_{i=j}^{n} \frac{1}{v_j}\right)^{-1} \quad \text{HarmonicMean}$$

$$F(v_1, \ldots, v_n) = \left(\frac{1}{n}\sum_{j=1}^{n} v_j^2\right)^{1/2} \quad \text{QuadraticMean}$$

These are special cases of the power mean

$$F(v_1, \ldots, v_n) = \left(\frac{1}{n}\sum_{j=1}^{n} v_j^r\right)^{1/r} \quad \text{for } r \neq 0$$

We see that if $r = 1$ we get the arithmetic mean, if $r = -1$ we get the harmonic and if $r = 2$ we get the quadratic.

A further generalization of the class of mean operators is the quasi-arithmetic means. Here if $g{:}R \to R$, called the generating function, is a continuous strictly monotone function then we define a quasi-arithmetic mean by

$$F(v_1, \ldots, v_n) = g^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} g(v_j)\right)$$

The preceding examples are special cases of these for different g. It is interesting to note that the geometric mean $F(v_1, \ldots, v_n) = \left(\prod_{j=1}^{n} v_j\right)^{1/n}$ has $g(x) = \log(x)$. Another example of quasi-arithmetic mean is the exponential mean here $g(x) = e^{\alpha x}$ for $\alpha \neq 0$ and in this case

$$F(v_1, \ldots, v_n) = \frac{1}{\alpha} \ln\left(\frac{1}{n}\sum_{j=1}^{n} e^{\alpha v_j}\right)$$

Each of these mean operators are aggregating the arguments in some different manner. Some kinds of relationships can be seen to hold with respect to these different aggregators [8]. Let us denote

$$F_g(v_1, \ldots, v_n) = g^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} g(v_j)\right)$$

$$F_h(v_1, \ldots, v_n) = h^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} h(v_j)\right)$$

We observe that if $g(x) = b\ \ h(x) + d$ and $b \neq 0$ then $F_h(v_1, \ldots, v_n) = F_g(v_1, \ldots, v_n)$. We also note that the two quasi–arithmetic means F_h and F_g satisfy $F_h(v_1, \ldots, v_n) \leqslant F_g(v_1, \ldots, v_n)$ if either the composition g h^{-1} is convex and g is decreasing or $g \cdot h^{-1}$ is concave and g is increasing. More specifically if g and h are power means $g(x) = x^\alpha$ and $h(x) = x^\beta$ and if $\alpha < \beta$ then

$$F_h(v_1, \ldots, v_n) \geqslant F_g(v_1, \ldots, v_n)$$

We note that two other important examples of means are the Max and Min operators

$$F(v_1, \ldots, v_n) = \text{Max}_j[a_j]$$
$$F(v_1, \ldots, v_n) = \text{Min}_j[a_j]$$

We observe that if $g(x) = x^\alpha$ and if $\alpha \to \infty$ then $F \to$ Max and if $\alpha \to -\infty$ then $F \to$ Min. We note that these are respectively the largest and smallest means, that is for any F we have

$$\text{Min}_j[v_j] \leqslant F(v_1, \ldots, v_n) \leqslant \text{Max}_j[v_j]$$

This follows from the boundedness requirement.

Closely related to these is the median operator, which is also a mean. We recall if n is odd then median f is defined as

$$F(v_1, \ldots, v_n) = v_{\left[\frac{(n+1)}{2}\right]}$$

where $v_{\left[\frac{(n+1)}{2}\right]}$ indicates the $\frac{n+1}{2}$ largest of the arguments. For the case, where n is even we can define

$$F(v_1, \ldots, v_n) = \frac{1}{2}\left(v_{\left[\frac{n}{2}\right]} + v_{\left[\frac{n}{2}+1\right]}\right)$$

where $v_{[k]}$ indicates the kth largest argument. We note that other possible definitions for the median in the case of even n, are,