Information Fusion 27 (2016) 189–197

Contents lists available at ScienceDirect

Information Fusion

journal homepage: www.elsevier.com/locate/inffus

# Construction of admissible linear orders for interval-valued Atanassov intuitionistic fuzzy sets with an application to decision making

L. De Miguel [a], H. Bustince [a,c,*], J. Fernandez [a], E. Induráin [b], A. Kolesárová [d], R. Mesiar [d,e,f]

[a] Departamento de Automática y Computación, Universidad Pública de Navarra, Campus Arrosadia s/n, 31006 Pamplona, Spain
[b] Departamento de Matemáticas, Universidad Pública de Navarra, Campus Arrosadia s/n, 31006 Pamplona, Spain
[c] Institute of Smart Cities, Universidad Publica de Navarra, Campus Arrosadia s/n, 31006 Pamplona, Spain
[d] Institute of Information Engineering, Automation and Mathematics, Slovak University of Technology, 81237 Bratislava, Slovakia
[e] Slovak University of Technology, Radlinskeho 11, Bratislava, Slovakia
[f] Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, 18208 Prague, Czech Republic

ARTICLE INFO

*Article history:*
Received 10 November 2014
Received in revised form 23 March 2015
Accepted 28 March 2015
Available online 4 April 2015

*Keywords:*
Interval-valued Atanassov intuitionistic fuzzy set
Interval linear order
Interval-valued Atanassov intuitionistic multi-expert decision making

ABSTRACT

In this work we introduce a method for constructing linear orders between pairs of intervals by using aggregation functions. We adapt this method to the case of interval-valued Atanassov intuitionistic fuzzy sets and we apply these sets and the considered orders to a decision making problem.


## 1. Introduction

In decision making problems it may happen that, after the exploitation phase, the best alternatives are equally ranked and it is not possible to decide which one is the best. It has been noticed [1] that these troubles often appear when the entries of the considered fuzzy preference matrix are close to 0.5, that is, when the experts have doubts about their preferences of some alternatives over the others. In this situation, the systematic use of extensions of fuzzy sets has been shown to be a really useful tool [2]. Among those fuzzy sets, interval-valued fuzzy sets (IVFSs) [3–5] or, equivalently, Atanassov intuitionistic fuzzy sets (AIFs) [6] play indeed a crucial role.

In some special cases, despite the fact of using IVFSs and AIFs, still remain problems that are similar to those encountered in the previous ones. For these new last situations we may use the interval-valued Atanassov intuitionistic fuzzy sets (IVAIFSs) [7]. Besides, the use of intervals to represent membership and non-membership has, from our point of view, a double advantage:

1. If we want to model environments where there exist non-comparable elements, it will be enough to use classical partial orders between intervals. This is not the case in this work.
2. If we must represent ignorance [8] associated to the datum given by an expert, we can understand the length of the intervals as a representation of such ignorance. If, in these cases, we need to be able to compare any two data, then we can use any of the linear orders we consider here.

Once the decision of using IVAIFSs to deal with a decision making problem has been reached, we should choose, accordingly, a linear order between pairs of intervals. In this way, we will select as the best option the alternative which is associated to the largest pair of intervals, with respect to the considered linear order.

Moreover, in decision making problems we must also aggregate the information furnished by the experts by means of aggregation functions [9–11].

All these considerations have led us to aim the following objectives:

(1) To use aggregation functions for building linear orders for pairs of intervals whose end-points belong to the unit interval.
(2) To study methods for constructing linear orders on the set of IVAIFSs.

* Corresponding author at: Departamento de Automática y Computación, Universidad Pública de Navarra, Campus Arrosadia s/n, 31006 Pamplona, Spain.

*E-mail addresses:* laura.demiguel@unavarra.es (L. De Miguel), bustince@unavarra.es (H. Bustince), fcojavier.fernandez@unavarra.es (J. Fernandez), steiner@unavarra.es (E. Induráin), anna.kolesarova@stuba.sk (A. Kolesárová), mesiar@math.sk (R. Mesiar).

http://dx.doi.org/10.1016/j.inffus.2015.03.004
1566-2535/

(3) To deal with the exploitation phase of decision making problems through IVAIFSs, by using the previously built linear orders.

The structure of this paper is the following. In Section 2 we introduce the notation and recall some well-known notions. In Sections 3,4, we construct two classes of linear orders between pairs of intervals. Section 5 contains an application of our theoretical results to group decision making. In particular, we provide two algorithms. Some concluding remarks as well as suggestions for further research close the paper.

## 2. Previous concepts and results

We start by recalling some well-known concepts that will be useful for subsequent developments throughout the paper.

### 2.1. On orders and partially ordered sets

Given a partially ordered set (poset) $(P, \preceq)$, we say that

(a) $1_P$ is the top of the poset if for all $x \in P$ it holds $x \preceq 1_P$.
(b) $0_P$ is the bottom of the poset if for all $x \in P$ it holds $0_P \preceq x$.

In case they exist, $1_P$ and $0_P$ are unique.

Let $K([0,1]) \subset \mathbb{R}^2$ be given by

$$K([0,1]) = \{(\underline{x}, \overline{x}) \in [0,1] \times [0,1] | \underline{x} \leqslant \overline{x}\},$$

and let $L([0,1])$ be the set of all closed subintervals of the unit interval, that is

$$L([0,1]) = \{\mathbf{x} | \mathbf{x} = [\underline{x}, \overline{x}] \text{ such that } 0 \leqslant \underline{x} \leqslant \overline{x} \leqslant 1\}.$$

There is a straightforward bijection $i : K([0,1]) \to L([0,1])$ given by $i((\underline{x}, \overline{x})) = [\underline{x}, \overline{x}] = \mathbf{x}$. Through this bijection, the partial order on $\mathbb{R}^2$, $(a,b) \leq_2 (c,d)$ if and only if $a \leqslant c$ and $b \leqslant d$ induces an equivalent partial order on $L([0,1])$, namely,

$$\mathbf{x} \preceq_2 \mathbf{y} \text{ iff } \underline{x} \leqslant \underline{y} \text{ and } \overline{x} \leqslant \overline{y}. \quad (1)$$

In this way, $(L([0,1]), \preceq_2)$ is a poset whose bottom and top are, respectively, $\mathbf{0} = [0,0]$ and $\mathbf{1} = [1,1]$. In fact, the bijection above is a lattice isomorphism.[1]

We refer as $(L([0,1]))^2$, to the universe of pairs of intervals, that is,

$$(L([0,1]))^2 = \{(\mathbf{x}, \mathbf{y}) = ([\underline{x}, \overline{x}], [\underline{y}, \overline{y}]) \text{ with } \underline{x}, \overline{x}, \underline{y}, \overline{y} \in [0,1]\}.$$

Similarly to what happens in the case of $\mathbb{R}^2$ and $L([0,1])$, the partial order on $\mathbb{R}^4$, given by $(a_1, b_1, c_1, d_1) \leq_4 (a_2, b_2, c_2, d_2)$ if and only if $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$ and $c_1 \leqslant c_2$ and $d_1 \leqslant d_2$, also induces an equivalent partial order $\preceq_4$ on $(L([0,1]))^2$, given by

$$(\mathbf{x_1}, \mathbf{y_1}) \preceq_4 (\mathbf{x_2}, \mathbf{y_2}) \text{ if and only if } \underline{x}_1 \leqslant \underline{x}_2 \text{ and } \overline{x}_1 \leqslant \overline{x}_2 \text{ and } \underline{y}_1 \leqslant \underline{y}_2 \text{ and } \overline{y}_1 \leqslant \overline{y}_2. \quad (2)$$

In this way, $((L([0,1]))^2, \preceq_4)$ becomes a poset whose bottom and top are, respectively, $(\mathbf{0}, \mathbf{0}) = ([0,0],[0,0])$ and $(\mathbf{1}, \mathbf{1}) = ([1,1],[1,1])$.

[1] This kind of sets, namely $K([0,1])$ and $L([0,1])$ have already been used, suitably equipped with some order and latticial structure [12,13], to construct some universal codomain where it was possible to represent different kinds of orderings as, e.g., total preorders, interval-orders and semiorders by means of a single function that preserves the ordinal structure. The bijection $i : K([0,1]) \to L([0,1])$ has also been considered in those approaches, and some other similar bijections and/or latticial isomorphism as well as order isotonies have also been introduced accordingly. By the way, another universal codomain to represent different kinds of orderings, which is essentially equivalent to $K([0,1])$, consists of triangular and symmetric fuzzy numbers. For further information see [14–17].

**Definition 2.1** [18]. An order $\preceq$ on $L([0,1])$ is said to be admissible if it is linear and refines the order $\preceq_2$, i.e., it is a linear order satisfying that for all $\mathbf{x}, \mathbf{y} \in L([0,1])$ such that $\mathbf{x} \preceq_2 \mathbf{y}$ it holds $\mathbf{x} \preceq \mathbf{y}$.

**Example 2.1.** The lexicographic orders on $L([0,1])$, given by

- $\mathbf{x} \preceq_{\text{lex}1} \mathbf{y}$ if and only if $(\underline{x} < \underline{y})$ or $(\underline{x} = \underline{y}$ and $\overline{x} \leqslant \overline{y})$ (lexicographic-1 order), and
- $\mathbf{x} \preceq_{\text{lex}2} \mathbf{y}$ if and only if $(\overline{x} < \overline{y})$ or $(\overline{x} = \overline{y}$ and $\underline{x} \leqslant \underline{y})$ (lexicographic-2 order), are admissible.

### 2.2. Extensions of fuzzy sets

**Definition 2.2** [6]. Let $U$ be a nonempty set usually called a universe. An Atanassov's Intuitionistic Fuzzy Set (AIFS) $F$ over $U$ is given by

$$F = \{\langle u, \mu_F(u), \nu_F(u) \rangle | u \in U\}$$

where $\mu_F : U \to [0,1]$ defines the membership degree of the element $u \in U$ to $F$ and $\nu_F : U \to [0,1]$ defines its nonmembership degree to the same set $F$. Besides, the functions $\mu_F$ and $\nu_F$ satisfy that, for all $u \in U$, $\mu_F(u) + \nu_F(u) \leqslant 1$.

The pair $(\mu_F(u), \nu_F(u))$ is called an intuitionistic pair, $\mathcal{L}([0,1])$ being the set of all possible intuitionistic pairs, i.e.,

$$\mathcal{L}([0,1]) = \{\mathbf{a} | \mathbf{a} = (a_1, a_2),\ a_1, a_2 \in [0,1] \text{ and } a_1 + a_2 \leqslant 1\}.$$

In [6], Atanassov introduced a partial order in the universe of AIFSs.

**Definition 2.3.** Let $F_1, F_2$ be two AIFSs. According to the order given by Atanassov in [6]

$$F_1 \leqslant F_2 \text{ if and only if for all } u \in U, \mu_{F_1}(u) \leqslant \mu_{F_2}(u) \text{ and } \nu_{F_1}(u) \geqslant \nu_{F_2}(u).$$

**Definition 2.4** [7]. Let $U$ be a universe. An Interval-Valued Atanassov Intuitionistic Fuzzy Set (IVAIFS) $G$ over $U$ is given by

$$G = \{\langle u, m_G(u), n_G(u) \rangle | u \in U\}$$

where $m_G : U \to L([0,1])$ defines the membership degree of the element $u \in U$ to $F$ and $n_G : U \to L([0,1])$ defines its nonmembership degree to the same universe $U$. Moreover, for all $u \in U$, the sum of the upper boundary values of $m_G(u)$ and $n_G(u)$ must be lower than or equal to 1.

The pair $(m_G(u), n_G(u))$ is called an interval-valued intuitionistic pair, being $\mathcal{L}_{IV}([0,1])$ the set of all possible interval-valued intuitionistic pairs, i.e.,

$$\mathcal{L}_{IV}([0,1]) = \{(\mathbf{x}, \mathbf{y}), \text{ with } \mathbf{x}, \mathbf{y} \in L([0,1]) \text{ and } \overline{x} + \overline{y} \leqslant 1\}.$$

**Remark 1.** Note that $\mathcal{L}_{IV}([0,1])$ consists of special types of intervals, while $(\mathcal{L}([0,1]))^2$ is a set of all possible intuitionistic pairs.

**Definition 2.5.** Let $G_1, G_2$ be two IVAIFSs. According to the order given by Atanassov in [7], $G_1 \preceq G_2$ if and only if, for all $u \in U$,

$$m_{G1}(u) \preceq_2 m_{G2}(u) \text{ and } n_{G2}(u) \preceq_2 n_{G1}(u),$$

where $\preceq_2$ is the partial order on $L([0,1])$ given in Eq. (1).