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Lexicographic ordinal OWA aggregation of multiple criteria

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ABSTRACT

We consider a special case of multi-criteria decision making. Here the information provided by the decision maker about an alternative's satisfaction to the criteria by provided in terms of values that only have an ordinal nature. Linguistic terms such as high, medium and low are examples of this type of valuation. In addition we consider the situation in which there exists a lexicographic based priority ordering among the criteria. This type of lexicographic order can arise in situations in which we want to limit the ability of the low priority criteria to compensate for lack of satisfaction to higher priority criteria. For example in selecting an automobile where safety and cost are our criteria a lexicographic ordering arises when we do not want to allow the fact that a car is extremely inexpensive to compensate for the fact that it may not be safe. To help in the construction of multi-criteria aggregation functions we introduce the concept of criteria organization to indicate the body of knowledge that guides how we combine an alternative's satisfactions to the individual criteria to obtain its overall satisfaction.

1. Introduction to ordinal multi-criteria aggregation

Criteria aggregation problems are prevalent in many applications [1–3]. They generally involve a collection of criteria, $C_1, \ldots, C_n$, and a set X of alternatives. Typically for each alternative $x \in X$ we are able to obtain a degree of satisfaction for each of the criteria, $C_i(x)$. The key step in this problem is the aggregation of the individual criteria satisfaction to obtain the overall satisfaction for an alternative [4–6]. We denote this overall satisfaction as

$$C(x) = \mathrm{Agg}(C_1(x), C_2(x), \ldots, C_n(x))$$

Once having $C(x)$ for each of the alternatives we are able to use this for the various tasks such selecting the best alternative. The use of the aggregation operation is often a surrogate for a more difficult problem. In this problem each alternative x is represented by a vector of criteria satisfactions $[C_1(x), \ldots, C_n(x)]$, and our objective is to compare or order these vectors. Generally the task of comparing n dimensional vectors is very difficult so an aggregation of the criteria satisfactions is used as a surrogate. In the light of this the properties and form of the aggregation function should be a reflection of the considerations that would play a role in the process of comparing the vectors.

One feature of this process of comparing these vectors is that if for all j we have $C_j(x) \geqslant C_j(y)$ then alternative y should not be preferred to alternative x. An implication of this is that a basic property desired of the aggregation operation is monotonicity. Specifically this requires that if $C_j(x) \geqslant C_j(y)$ for j all then $C(x) \geqslant C(y)$. Other properties are sometimes imposed but they are effectively defining how the aggregation is to be performed.

Fundamental to the construction of multi-criteria aggregation functions is what we call the organization of the criteria. By this we meant to indicate the body of knowledge that guides how we combine an alternative's satisfactions to the individual criteria to obtain its overall satisfaction. Since there exists a wide variety of ways that a collection of criteria can be organized there exists many different types of information that can be used to express our knowledge of the criteria organization in a given situation. One example is symmetry which indicates that all the criteria are treated in the same way. Another example is what we referred to as the scope of the criteria organization [7]. By this we meant information about how many of the criteria we desire to be satisfied. For example given a list of criteria can we just satisfy some of them or are we required to satisfy all of them.

Here we consider the case of a lexicographic ordering between the criteria. This type of relationship formally involves a priority relationship between the criteria. An example of this kind of relationship can arise in case of selecting a bicycle for a child. Here we may have two criteria safety and cost. Typically in this kind a decision we may for allow some kind of tradeoff between the criteria. That is we may allow a good deal in price to compensate a lack of safety. If a lexicographic relationship exists in which safety has priority over price we do not allow very good satisfaction to the criteria of cost to compensate for lack of satisfaction to the criteria of safety. We need satisfaction to the safety criteria before we can consider any impact of the cost criteria. In this work we suggest a method for formally implementing this type of relationship by

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making the importances of the lower priority criteria depend on the satisfactions of the higher priority criteria.

In many real applications of multi-criteria decision making the measure of an alternative's satisfaction to criteria by provided by a human informant is at best expressed in terms of values that only have an ordinal nature. Linguistic terms such as high, medium and low are examples of this type of valuation. In the light of this reality an ability to aggregate ordinal information is useful.

A typical example of an ordinal scale is $D = \{d_0, d_1, \ldots, d_q\}$ where the only structure is that $d_i > d_j$ if $i > j$. While using an ordinal scale generally eases the burden on the information provider it limits the operations available for the formulation of the function Agg. Three basic operators are available on an ordinal scale, join (Max/$\vee$), meet (Min/$\wedge$) and complement (negation). In particular if a and $b \in D$ and $a \geqslant b$ we have Max(a, b) = a and Min(a, b) = b. Finally we have negation. If $a = d_k$ then the negation of a neg(a) or $\bar{a} = d_{q-k}$. It is essentially order reversal. Generally we must construct our aggregation function using these operations.

Thus in this work we shall consider the construction of multi-criteria decision functions which can model a lexicographic relation between criteria in situations in the measures of satisfaction are drawn from an ordinal scale.

2. Preliminaries on ordinal aggregation

In [8] we introduced a class of ordinal aggregation operator called ordinal OWA operators of dimension n. Assume a_j, for j = 1–n, are a collection of ordinal values from D. Let Q: $\{1, 2, \ldots, n\} \to D$ be a mapping that associates with each j = 1–n a value $Q(j) \in D$ so that $Q(i) \geqslant Q(j)$ if $i > j$ and $Q(n) = d_q$. Using this we define the ordinal OWA operator as

$$\mathrm{Agg}(a_1, \ldots, a_n) = \operatorname*{Max}_{j=1}^{n}[Q(j) \wedge b_j]$$

where b_j is the jth largest of the a_i. If we let ind be a permutation so that ind(j) is the index of the jth largest a_i then $b_j = a_{\mathrm{ind}(j)}$ and we can express this ordinal OWA operator as

$$\mathrm{Agg}(a_1, \ldots, a_n) = \bigvee_{j=1}^{n}[Q(j) \wedge \ a_{\mathrm{ind}_{(j)}}]$$

The mapping Q provides a parameterization of this aggregation operator. Some special cases of Q are worth noting. If $Q(1) = d_q$ then $\mathrm{Agg}(a_1, \ldots, a_n) = \mathrm{Max}_i[a_i]$. If $Q(n-1) = d_0$ then $Q(j) = d_0$ for all $j \neq n$ and $\mathrm{Agg}(a_1, \ldots, a_n) = \mathrm{Min}_i[a_i]$. Intermediate to these is a case when $Q(j) = d_0$ for $j < k$ and $Q(j) = d_q$ for $j \geqslant k$ in this case $\mathrm{Agg}(a_1, \ldots, a_n) = b_k$. The median is a special case of this where $k = \frac{n+1}{2}$ if n is odd otherwise $k = \frac{n}{2}$. Another special case is the following. Assume $\alpha \in D$ and let $Q(j) = \alpha$ for $j < n$ and $Q(n) = d_q$ in this case we get a kind of α weighted average $\mathrm{Agg}(a_1, \ldots, a_n) = (\alpha \wedge \mathrm{Max}_i[a_i]) \vee \mathrm{Min}_i[a_i]$.

We observe that this ordinal OWA operator can be used for the aggregation of criteria satisfaction in the ordinal domain, $C(x) = \mathrm{Agg}[C_1(x), \ldots, C_n(x)] = \bigvee_{j=1}^{n}[Q(j) \wedge C_{\mathrm{ind}_{(j)}}(x)]$ where $C_{\mathrm{ind}(j)}(x)$ is the jth most satisfied criteria.

We see that the choice of Q determines the type aggregation to be performed. We must investigate the connection between the choice of Q and the type of aggregation desired. As we shall subsequently see the choice of Q can be used to reflect information about the scope of the organization of the criteria.

We now suggest an approach for obtaining the ordinal OWA operator which will allow us to interpret the meaning of the different forms for Q. We again let $C = \{C_1, \ldots, C_n\}$ be a collection of criteria and we let μ be a monotonic set measure on $C, \mu : 2^C \to D$, having the properties:

$$(1)\ \mu(\varnothing) = d_0, \quad (2)\ \mu(C) = d_q \quad \text{and} \quad (3)\ \mu(A) \geqslant \mu(B) \quad \text{if } B \subseteq A.$$

Here we interpret μ so that for any subset A of C the term $\mu(A)$ indicates the degree of acceptability of a solution that satisfies all the criteria in A. Here then μ is providing information about the organization of the criteria.

Let A be any subset of C. For any alternative x we can express the degree to which x satisfies all the criteria in A as $A(x) = \mathrm{Min}_{C_i \in A}[C_i(x)]$. It is the degree of satisfaction of the least satisfied criteria in A.

We can now express the overall satisfaction by alternative x to the set of criteria C under the criteria organization expressed by μ. Here we need to find some subset of C that is satisfied by x and is an acceptable subset. We can express this as

$$C_\mu(x) = \operatorname*{Max}_{A \subseteq C}[\mu(A) \wedge A(x)]$$

Here we are letting $A(x)$ indicate the degree to which x satisfies all criteria in A, $\mathrm{Min}_{C_i \in A}[C_i(x)]$.

We now let ind be an index function so that ind(j) is the index of the jth largest $C_i(x)$. We further note that for any $A \neq \varnothing$ the value of $A(x)$ will be equal to $C_{\mathrm{ind}(j)}(x)$ for some j = 1–n. We now shall partition $2^C - \{\varnothing\}$ into n subsets, $C^1(x), \ldots, C^j(x), \ldots, C^n(x)$ so that a subset A is in $C^j(x)$ if $A(x) = C_{\mathrm{ind}(j)}(x)$, the jth most satisfied criteria under x. Using this we can express

$$C_\mu(x) = \bigvee_{j=1}^{n}(\mathrm{Max}_{A \in C^j}[\mu(A) \wedge C_{\mathrm{ind}(j)}(x)])$$
$$= \bigvee_{j=1}^{n}(C_{\mathrm{ind}(j)}(x) \wedge \mathrm{Max}_{A \in C^j}[\mu(A)])$$

Let us consider the subset $C^j(x)$. We see that a subset A of C is in $C^j(x)$ if it meets two conditions: it must contain the criteria $C_{\mathrm{ind}(j)}(x)$ and it must <u>not</u> contain any criteria $C_{\mathrm{ind}(k)}(x)$ for $k > j$. Consider the subset of criteria $H_j(x) = \{C_{\mathrm{ind}(k)}(x)/k \leqslant j\}$. First we note that $H_j(x) \in C^j(x)$ and in addition any set $A \in C^j(x)$ must be such that $A \subseteq H_j(x)$. From the monotonicity condition associated with μ we can therefore conclude that $\mathrm{Max}_{A \in C^j}[\mu(A)] = \mu(H_j(x))$ and therefore

$$C_\mu(x) = \bigvee_{j=1}^{n}(C_{\mathrm{ind}(j)}(x) \wedge \mu(H_j(x)))$$

Here $H_j(x)$ as defined above is the subset of criteria with the j largest satisfactions under x. We note that $\mu(H_j(x))$ is the degree of acceptability of just trying to satisfy the subset of criteria in $H_j(x)$.

Let us now consider a special case of μ. Here we shall let $\mu(A)$ just depend on the number of criteria in A. In particular $\mu(A) = Q(|A|)$ where $|A|$ is the cardinality of A and Q is a function from $N = \{0, \ldots, n\}$ to D. Because of the properties of μ we see that Q must have some comparable properties: (1) $Q(0) = d_0$, (2) $Q(n) = d_q$ and (3) $Q(j) \geqslant Q(k)$ if $j \geqslant k$.

Let us now return to our formula $C_\mu(x) = \mathrm{Max}_{j=1-n}(C_{\mathrm{ind}(j)}(x) \wedge \mu(H_j(x)))$. Since $H_j(x) = \{C_{\mathrm{ind}(k)}/\text{for } k \leqslant j\}$ then H_j always has j elements and hence $\mu(H_j(x)) = Q(j)$ and therefore

$$C_\mu(x) = \bigvee_{j=1}^{n}(C_{\mathrm{ind}(j)}(x) \wedge Q(j))$$

The above formula is what we called the ordinal OWA operator. So we see that in the case of the ordinal OWA operator the term $Q(j)$ is the degree of acceptability for satisfying any j criteria. Thus we see that $Q(j)$ is a reflection of the organization of the criteria.

We have suggested a general formulation for determining the overall satisfaction of an alternative to collection of criteria in a situation in which we have a measure μ indicating the acceptability of different subsets of criteria. In the preceding we considered the special case where $\mu(A)$ is just a function of the number of criteria in A, the cardinality of A. In the following we shall consider an