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Monometrics and their role in the rationalisation of ranking rules

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ABSTRACT

The aggregation of rankings is a long-standing problem that consists of, given a profile of rankings, obtaining the single ranking that best represents the nature of this given profile. Under the name of metric rationalisation of ranking rules, it has been proven that most ranking rules can be characterized as minimizing the distance to a consensus state for some appropriate distance function. In this paper, we propose to consider monometrics instead of distance functions. Although these concepts are closely related, monometrics better capture the nature of the problem, as the purpose of a monometric is to preserve a given betweenness relation. This is obviously only meaningful when an interesting betweenness relation is fixed, for instance, the one based on reversals in rankings proposed by Kemeny. In this way, ranking rules can be characterized in terms of a consensus state and a monometric.

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1. Introduction

Aggregation of rankings has been a relevant matter of study since the eighteenth century. It consists of, given a list of rankings (from here on called profile of rankings), obtaining the single ranking that best represents the nature of this given profile. This ranking is considered the 'winner' by the aggregation method (from here on called ranking rule). The aggregation of rankings has been addressed in many scientific disciplines, including medicine [1], consumer preference analysis [2], computer science [3], management science [4] and social choice theory [5,6]. Nevertheless, due to the natural interpretation of the aggregation of rankings as a voting procedure, social choice theory is considered the most prominent field of application.

Many ranking rules have been proposed since the eighteenth century, when the works of Rousseau [7], Borda [8] and Condorcet [9] laid the foundations of social choice theory. In a recent paper, Meskanen and Nurmi [10] analysed the notion of consensus state (or simply consensus). In general, a profile is said to be in a consensus state when determining a winning ranking is obvious. A trivial consensus state is unanimity [11], where each voter has the exact same preferences. Another slightly more involved one, is the existence of a Condorcet ranking [9]. Several authors, such as Nitzan [12], Lerer and Nitzan [13], Campbell and Nitzan [14] and Meskanen and Nurmi [10,15], have advocated that ranking rules can be characterized as minimizing the distance to a consensus state for some appropriate distance function. This characterization is known as metric rationalisation of ranking rules. A well-known method of this type is that of Kemeny [11], where the Kendall (tau) distance [16] between rankings and the unanimous consensus state are considered. Other relevant proposals are due to Bogart [17,18] or Cook and Seiford [19,20], where different distance functions are considered; or due to Meskanen and Nurmi [10,21], Rademaker and De Baets [22] or Pérez-Fernández et al. [23], where different consensus states are considered [10,15,24–26].

Perhaps due to the inaccurate term 'closeness' used to define the problem, where the 'closest' profile of rankings in the required consensus state is searched, researchers have focused too strongly on the notion of distance function. Nevertheless, the nature of the problem requires the use of another closely related type of function. Distance functions are actually too restrictive. Note that, instead of looking for the 'closest' profile of rankings in the required consensus state, we are interested in minimizing the cost (loss) of changing the given profile into another one in the required consensus state. As the converse is not required, there is no need to ask for symmetry. In addition, this cost might not be (sub-)additive (for instance when considering a penalized distance function); hence, there is also no need to ask for the triangle inequality. On the other hand, a notion of betweenness between profiles of rankings may describe the conditions under which a profile should be 'closer' to the original one than another profile. After all, if a profile $\mathscr{R}'$ is in between the original profile $\mathscr{R}$ and another profile $\mathscr{R}''$, then the cost of changing $\mathscr{R}$ into $\mathscr{R}'$ should be lower than that of changing $\mathscr{R}$ into $\mathscr{R}''$. A function satisfying the latter axiom and, at the

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same time, the non-negativity and coincidence axioms, is called a monometric. These monometrics will extend the analysis of ranking rules from the point of view of searching for a consensus state.

The rest of the paper is organized as follows. Section 2 is devoted to the analysis of monometrics. In Section 3, the calculation of the cost is presented as an optimization problem. A procedure for hierarchically combining monometrics is proposed in Section 4. The characterization of ranking rules in terms of a consensus state and a monometric is addressed in Section 5. We end with some conclusions and open problems in Section 6.

2. Monometrics

In this section, we introduce the concept of a monometric, which is closely related to that of a distance function or metric. Like a distance function, a monometric satisfies the axioms of non-negativity and coincidence, but a monometric requires compatibility with a given betweenness relation [27] and does not impose symmetry nor the triangle inequality.

2.1. General case

The notion of distance function or metric is a well-known concept in mathematics.

Definition 1. A function $d : A \times A \to \mathbb{R}$ is called a distance function (on the set A) if it satisfies the following four properties:

(i) **Non-negativity**: for any $a, b \in A$, it holds that

$$d(a,b) \geq 0 .$$

(ii) **Coincidence**: for any $a, b \in A$, it holds that

$$d(a,b) = 0 \;\Leftrightarrow\; a = b .$$

(iii) **Symmetry**: for any $a, b \in A$, it holds that

$$d(a,b) = d(b,a) .$$

(iv) **Triangle inequality**: for any $a, b, c \in A$, it holds that

$$d(a,c) \leq d(a,b) + d(b,c) .$$

A betweenness relation is a ternary relation, introduced by Pasch [27] and further developed by Huntington and Kline [28], that describes when an element is in between two other ones. In what follows, we adhere to the formal relaxed definition given by Pitcher and Smiley [29], requiring a minimal set of axioms. Actually, they also proposed additional axioms concerning transitivity. Further additional axioms have been proposed in literature [27,28], [30].

Definition 2. A ternary relation R on a set A is called a betweenness relation if it satisfies the following two properties:

(i) **Symmetry in the end points**: for any $a, b, c \in A$, it holds that

$$(a,b,c) \in R \quad \Leftrightarrow \quad (c,b,a) \in R .$$

(ii) **Closure**: for any $a, b, c \in A$, it holds that

$$\big((a,b,c) \in R \wedge (a,c,b) \in R\big) \quad \Leftrightarrow \quad b = c .$$

The formula '$(a, b, c) \in R$' is read as 'b is in between a and c' and is denoted as $[a, b, c]$ when no confusion is possible.

Note that although no transitivity axioms are required in this paper, they are necessary conditions in order to guarantee the existence of an order relation $\leq$ that agrees with R, i.e. for which it holds that $(x, y, z) \in R$ if and only if $x \leq y \leq z$ or $z \leq y \leq x$. For further details about the relationship between order relations and betweenness relations, we refer to [30].

After fixing a betweenness relation, monometrics can be introduced, which are functions satisfying the non-negativity and coincidence axioms of a distance function, while preserving the given betweenness relation.

Definition 3. Let A and B be two sets such that $A \subseteq B$ and let R be a betweenness relation on B. A function $M : A \times B \to \mathbb{R}$ is called a monometric (w.r.t. R) if it satisfies the following three properties:

(i) **Non-negativity**: for any $a \in A$ and any $b \in B$, it holds that

$$M(a,b) \geq 0 .$$

(ii) **Coincidence**: for any $a \in A$ and any $b \in B$, it holds that

$$M(a,b) = 0 \;\Leftrightarrow\; a = b .$$

(iii) **Compatibility**: for any $a \in A$ and any $b, c \in B$ such that $[a, b, c]$, it holds that

$$M(a,b) \leq M(a,c) .$$

In case the sets A and B coincide, we say that M is a monometric on A.

For any $a \in A$ and any $b \in B$, $M(a, b)$ is called the cost of changing a into b. The set A is called the set of observable elements and the set B is called the set of reachable elements. It may be the case that, after changing an observable element into a reachable one, we can no longer recognize whether or not the latter element belongs to the set of observable elements (see [23] for an example of non-characterizable set of reachable elements).

Note that, by considering an appropriate betweenness relation, every distance function can be considered a monometric. In the following example, two generic betweenness relations are proposed w.r.t. which every distance function is a monometric.

Proposition 1. *A distance function $d : A \times A \to \mathbb{R}$ (on the set A) is a monometric w.r.t. both of the following betweenness relations:*

(i) $R_1 = \{(a,b,c) \in A^3 \mid a = b \vee b = c\}$;
(ii) $R_2 = \{(a,b,c) \in A^3 \mid d(a,c) = d(a,b) + d(b,c)\}$.

Proof. We first prove that both R_1 and R_2 satisfy the two axioms of a betweenness relation (on A). Symmetry in the end points: for any $a, b, c \in A$, it holds that

$$\begin{aligned}(a,b,c) \in R_1 &\Leftrightarrow (a=b) \vee (b=c)\\ &\Leftrightarrow (c=b) \vee (b=a)\\ &\Leftrightarrow (c,b,a) \in R_1\end{aligned}$$

and

$$\begin{aligned}(a,b,c) \in R_2 &\Leftrightarrow d(a,c) = d(a,b) + d(b,c)\\ &\Leftrightarrow d(c,a) = d(b,a) + d(c,b)\\ &\Leftrightarrow (c,b,a) \in R_2,\end{aligned}$$

due to the symmetry of d. Closure: for any $a, b, c \in A$, it holds that

$$\begin{aligned}&\big((a,b,c) \in R_1\big) \wedge \big((a,c,b) \in R_1\big)\\ &\Leftrightarrow (a = b \vee b = c) \wedge (a = c \vee b = c)\\ &\Leftrightarrow b = c\end{aligned}$$

and

$$\begin{aligned}&\big((a,b,c) \in R_2\big) \wedge \big((a,c,b) \in R_2\big)\\ &\Leftrightarrow d(a,c) = d(a,b) + d(b,c) \wedge d(a,b) = d(a,c) + d(c,b)\\ &\Leftrightarrow d(b,c) = 0\\ &\Leftrightarrow b = c.\end{aligned}$$

Next, we prove that d satisfies the three axioms of a monometric (on A) w.r.t. both R_1 and R_2. The non-negativity and coincidence

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