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Numerical integration for the Choquet integral

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ABSTRACT

Choquet integrals with respect to non-additive (or fuzzy measures) have been used in a large number of applications because they permit us to integrate information from different sources when there are interactions. Successful applications use a discrete reference set.

In the case of measures on a continuous reference set, as e.g. the real line, few results have been obtained that permit us to have an analytical expression of the integral. However, in most of the cases there is no such analytical expression.

In this paper we describe how to perform the numerical integration of a Choquet integral with respect to a non-additive measure.

1. Introduction

The Choquet integral [1] is an integral that can be used to integrate a function with respect to a non-additive measure. It is well known that this integral generalizes the Lebesgue integral in the case that the measure is additive.

A large number of applications [2,3] exist nowadays mainly in the areas of decision making, computer science (e.g., computer vision [4], fuzzy reasoning [5,6]) and economics. It has also been used to build models of decision under uncertainty and risk.

Most existing applications typically use functions defined on a finite set of reference. This is the case of the works mentioned above and e.g. [7]. For functions on a finite set the computation of the Choquet integral is easy from a computational point of view.

This is not the case when the reference set is infinite. For example, if we want to integrate a function f: [0, 1] → [0, 1] with respect to a non-additive measure. In recent papers [8–12], a few results on analytic solutions of the Choquet integral for infinite reference sets have been obtained. These papers give analytical expressions for some families of functions and measures. We will review some of these results below.

Nevertheless, these results are rather limited and there is interest on computing the integral for arbitrary functions and arbitrary measures. For example, the Choquet integral is used in economics with continuous distorted probabilities (weighted functions combined with probabilities) for some risk models. The work in [13] also discusses some examples in a continuous setting in order to compute distances between non-additive measures. For some examples computation was not included because the analytical expressions for the Choquet integral are not available.

In this paper we consider the problem of computing a numerical Choquet integral. We present some algorithms for some families of functions and give some examples. Implementations have been done in the statistics software R. We compare the numerical integration of a monotonic function and the analytic expression that can be obtained using the results in [10] and [11] when the analytical expression can be found.

The structure of this paper is as follows. In Section 2 we present the results on non-additive measures and integrals which are needed in the rest of the paper. This section includes also some new results. In Section 3 our approach to numerical integration is presented and examples are given. The paper finishes with some conclusions and lines for future work.

2. Preliminaries

This section reviews the main topics needed in the rest of the paper. We first focus on the definitions of non-additive measures and the Choquet integral and latter we review some of the results for obtaining analytical solutions of the Choquet integral. This section finishes with some results on the continuity of distorted probabilities.

2.1. Non-additive measures and the Choquet integral

Non-additive measures are known with different names in the literature. Fuzzy measures, capacities, monotonic measures and

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monotonic games are some of them. A non-additive measure is a monotonic set function. We give its definition below. For details on fuzzy measures and integrals, definitions of measurable space and measurable function and related topics see e.g. [14–16].

Definition 1. Let $(X, \mathcal{F})$ be a measurable space. A set function μ defined on $\mathcal{F}$ is called a non-additive measure if an only if

- $0 \le \mu(A) \le \infty$ for any $A \in \mathcal{F}$;
- $\mu(\emptyset) = 0$;
- If $A_1 \subseteq A_2 \subseteq \mathcal{F}$ then $\mu(A_1) \le \mu(A_2)$.

Naturally, all additive measures satisfy the conditions of non-additive measures. So, the latter are a generalization of the former measures. Recall that a measure μ is additive when $\mu(A \cup B) = \mu(A) + \mu(B)$ for $A \cap B = \emptyset$.

Distorted Lebesgue measures and distorted probabilities are examples of non-additive measures.

First note that the Lebesgue measure λ for an interval $[a, b]$ is defined by $\lambda([a, b]) = b - a$, and that given a distortion function m which is increasing (or non-decreasing) and such that $m(0) = 0$, the measure $\mu(A) = m(\lambda(A))$ is a distorted Lebesgue measure. We denote a Lebesgue measure with distortion m by $\mu = \mu_m$. When P is a probability, $\mu(A) = m(P(A))$ is a distorted probability. In this case it is usual to require that $m(1) = 1$. We will use also the notation $m \circ \lambda$ and $m \circ P$.

Given a non-additive measure its conjugate is of interest.

Definition 2. Given a non-additive measure μ on $(X, \mathcal{F})$, with $\mu(X)$ finite (i.e., $\mu(X) < \infty$, we define its conjugate or dual as the measure μ^c defined as follows:

$$\mu^c(A) = \mu(X) - \mu(X \setminus A).$$

When a measure μ is additive, it holds that $\mu(A) = \mu^c(A)$.

Given a non-additive measure we can consider the problem of integrating a function with respect to this measure. The Choquet integral permits us to do so. This integral corresponds to the Lebesgue integral when the measure is σ-additive.

Definition 3 ([1]). Let X be a reference set, let $(X, \mathcal{A})$ be a measurable space, let μ be a non-additive measure on $(X, \mathcal{A})$, and let g be a measurable function $g : X \to [0, +\infty)$; then, the Choquet integral of g with respect to μ is defined by

$$(C)\int g d\mu := \int_0^\infty \mu_g(r) dr, \quad (1)$$

where $\mu_g(r) := \mu(\{x | g(x) > r\})$.

When the function g can take negative values, the functional above cannot be applied. Two alternative definitions have been considered in the literature, which are known as the symmetric (or Šipoš [17]) and asymmetric integral. Their definitions follow.

Definition 4. Let $(X, \mathcal{A})$ be a measurable space, let μ be a finite non-additive measure on $(X, \mathcal{A})$, and let g be a measurable function; then, the symmetric Choquet integral, also known as the Šipoš integral) of g with respect to μ is defined by

$$(S)\int g d\mu := (C)\int (g \vee 0) d\mu - (C)\int ((-g) \vee 0) d\mu. \quad (2)$$

Definition 5. Let $(X, \mathcal{A})$ be a measurable space, let μ be a non-additive measure on $(X, \mathcal{A})$, and let g be a measurable function; then, the asymmetric Choquet integral of g with respect to μ is defined by

$$(AC)\int g d\mu := (C)\int (g \vee 0) d\mu - (C)\int ((-g) \vee 0) d\mu^c, \quad (3)$$

where μ^c is the conjugate of μ.

In this work it is also relevant the Choquet integral of g with respect to a non-additive measure μ on a set A. This is defined by:

$$(C)\int_A g d\mu := \int_0^\infty \mu(\{x | g(x) > r\} \cap A) dr. \quad (4)$$

2.2. Some analytical solutions for the Choquet integral

The construction of analytical solutions for the Choquet integral has been studied in several recent papers. See e.g. [8–12].

We review below some results that are used to evaluate the methods for numerical integration proposed in this work.

We begin with results from [10].

Theorem 1 (Theorem 1 in [10]). *Let $f : [0, 1] \to \mathbb{R}$ be monotonic increasing with $f(0) = 0$ and differentiable. Let λ be the Lebesgue measure. Let m be defined as $m(x) = \sum_{k=1}^{s} a_k x^k$ for $a_k \in \mathbb{R}$, and let μ be a distorted Lebesgue measure $\mu = m \circ \lambda$. Let $\{f_k\}$ be the sequence of functions defined by*

$$f_1 = \int_0^x f d\lambda$$

$$f_{k+1} = \int_0^x f_k d\lambda$$

for $x \in [0, 1]$, $k = 1, 2, \ldots$. Then, we have

$$(C)\int_{[0,x]} f d\mu = \sum_{k=1}^{s} k! a_k f_k(x)$$

for $x \in [0, 1]$.

In addition, the following result was also proven.

Proposition 1 (Corollary 1 in [10]). *Let f be a continuous function f: $[0, 1] \to [0, 1]$ with $f(0) = 0$, and $\max_{x \in [0,1]} f(x) = 1$ and $\{x | f(x) = 1\}$ a finite set and λ be the Lebesgue measure on [0,1]. Then, there exists a monotone increasing function f^*: $[0, 1] \to [0, 1]$ such that*

$$(C)\int f^* dm \circ \lambda = (C)\int f dm \circ \lambda.$$

We review now another result from [11] which permits us to compute the Choquet integral analytically of other functions when the non-additive measure is a distorted Lebesgue measure.

Let $m(t)$, $g(t)$ and $f(t)$ be continuously differentiable. Let $\mu([\tau, t])$ be differentiable with respect to τ on $[0, t]$ for every $t > 0$. We require the regularity condition that $\mu(\{t\}) = 0$ holds for every $t \ge 0$. Let $\mu'([\tau, t])$ denote $(\partial/\partial\tau)\mu([\tau, t])$, where we note that $\mu'([\tau, t]) \le 0$ for $\tau \le t$. If μ is a distorted Lebesgue measure μ_m, then $\mu'([\tau, t]) = -m'(t - \tau)$ where $m'(t) = dm(t)/dt$.

Theorem 2 (Theorem 1 in [11]). *Let $\mathcal{F}^+$ be the class of measurable, non-negative, continuous and increasing (non-decreasing) functions such that $g : \mathbb{R}^+ \to \mathbb{R}^+$. Let $g \in \mathcal{F}^+$, then the Choquet integral of g with respect to μ on $[0, t]$ is represented as:*

$$(C)\int_{[0,t]} g d\mu = \int_0^\infty \mu(\{\tau | g(\tau) \ge r\} \cap [0, t]) dr = -\int_0^t \mu'([\tau, t]) g(\tau) d\tau, \quad (5)$$

and when the measure is a distorted Lebesgue measure $\mu = \mu_m$ then

$$\int_0^\infty \mu(\{\tau | g(\tau) \ge r\} \cap [0, t]) dr = \int_0^t m'(t - \tau) g(\tau) d\tau.$$

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a function. Then, its Laplace transformation is denoted by $H(s)$ and its inverse Laplace transformation by $h(t) = L^{-1}[H(s)]$.