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Multiphase image segmentation via equally distanced multiple well potential

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ABSTRACT

Variational models for image segmentation, e.g. Mumford–Shah variational model [47] and Chan–Vese model [21,59], generally involve a regularization term that penalizes the length of the boundaries of the segmentation. In practice often the length term is replaced by a weighted length, i.e., some portions of the set of boundaries are penalized more than other portions, thus unbalancing the geometric term of the segmentation functional.

In the present paper we consider a class of variational models in the framework of Γ-convergence theory. We propose a family of functionals defined on vector valued functions that involve a multiple well potential of the type arising in diffuse-interface models of phase transitions. A potential with equally distanced wells makes it possible to retrieve the penalization of the true (i.e., not weighted) length of the boundaries as the Γ-convergence parameter tends to zero. We explore the differences and the similarities of behavior of models in the proposed class, followed by some numerical experiments.

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1. Introduction

Image segmentation consists in looking for a partition of the spatial domain of an image into homogeneous regions, which helps to identify meaningful parts of objects in the image. Such a problem has been extensively studied by various approaches, such as mixture random-field models [28], Monte-Carlo Markov chain models [58], using graph-cut and spectral methods [55], and the Mumford–Shah variational image model [47]. In this paper, we focus on variational methods and on the approach based on Γ-convergence theory.

Let $\Omega \subset \mathbb{R}^2$ denote a bounded, open set with Lipschitz boundary, which represents the image domain, and let $f : \Omega \longrightarrow \mathbb{R}$ denote a bounded function which represents a given image. The Mumford–Shah variational approach to image segmentation consists in looking for a pair (u, K), with $K \subset \overline{\Omega}$ closed set and $u \in \mathcal{C}^1(\Omega \setminus K)$, which minimizes the functional

$$\mathcal{E}_{ms}(u,K) = \alpha \int_{\Omega\setminus K} |\nabla u|^2\, dx + \mathcal{H}^1(K) + \lambda \int_{\Omega} (u-f)^2\, dx.$$

Here $\mathcal{H}^1(K)$ denotes the 1-dimensional Hausdorff measure of the set K (the length if K is regular enough), and $\alpha,\ \lambda$ are positive weights. The function u is a denoised, piecewise smooth, approximation of the given image f, and the set K is the set of discontinuities of the approximate image u. The set K also constitutes the set of edges of the image which are then linked together according to a minimum length criterion. The piecewise constant Mumford–Shah model consists in minimizing the functional $\mathcal{E}_{ms}$ with respect to functions u which satisfy the further requirement $\nabla u(x) = 0$ in $\Omega \setminus K$.

The Chan–Vese model [21,59] is well-known with a successful level set implementation of piecewise constant Mumford–Shah variational problem:

$$\mathcal{E}_{cv}(u,K) = \mathcal{H}^1(K) + \lambda \sum_{i=1}^{N} \int_{\Omega_i} (u-f)^2\, dx. \tag{1}$$

Here N is a given integer and $\Omega_i,\ i = 1, \ldots, N$, are open subsets that constitute a Borel partition of Ω. The set $K \cap \Omega$ is the union of the part of the boundaries of the Ω_i inside Ω:

$$K \cap \Omega = \bigcup_{i=1}^{N} \partial\Omega_i \cap \Omega, \qquad \Omega = (K \cap \Omega) \bigcup_{i=1}^{N} \Omega_i. \tag{2}$$

For a fixed set K, the functional $\mathcal{E}_{cv}$ is minimized with respect to the function u by setting u equal to the mean value of f in Ω_i, for any

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$i = 1, \dots, N$. The subsets Ω_i are called phases of $\Omega \setminus K$. The segmentation is usually called two-phase when $N = 2$ and multiphase when $N > 2$. Then a local minimum of the functional is looked for by considering Euler–Lagrange equations which are solved by the level set method. This model has been extended to various settings including piecewise smooth Mumford–Shah variational model, vectorial multi-channel and texture segmentation.

In multiphase setting, the objective is to identify multiple different phases by keying the intensity discontinuities. There are several further region-based multiphase segmentation models introduced such as [6,23,34,42,57,59]. Brown et al. [16] considered convex formulation of the Chan–Vese segmentation, and Chambolle et al. [20] discussed a convex relaxation for a family of problems of minimal length partitions with interesting results. There are approaches using fuzzy region computation [32,33,53,39,38], other related work can be found at [19,27,40,41,37].

There are a number of models in image analysis using the diffuse-interface model and well-potential by making the connection with phase transition models. Ambrosio and Tortorelli [2,3] first approximated the Mumford–Shah functional by means of quadratic, elliptic functionals using the theory of Γ-convergence. This approximation has been numerically implemented for instance in [43] and a related approach has been studied in [44]. Shen [54] proposed using Modica and Mortola's phase transition model [45,46] for ownership distribution for a stochastic-variational model for soft Mumford–Shah segmentation. In [25], the authors considered an approximation of the 2.1D Sketch model proposed for segmentation with depth by Nitzberg et al. [48]. There are some well-potential approach on image inpainting such as [13,14,17,26,30]. Another interesting approach based on topology optimization is proposed in [4], where the authors introduce a new model to approximate the length of boundaries in an optimal partition problem.

We explore a class of multiphase segmentation models based on a Γ-convergence approach. We extend to a vectorial setting the multiphase model such as [34], which is based on the Γ-convergence result by Modica and Mortola [45,46]:

$$E_\epsilon(z) = \int_\Omega \left[\epsilon|\nabla z|^2 + \frac{1}{\epsilon}\sin^2(\pi z)\right] \mathrm{d}x + \lambda \sum_{k=0}^{N-1} \int_\Omega (c_k - f)^2 \mathrm{sinc}^2(z-k)\mathrm{d}x, \quad (3)$$

where N is a given integer, $-1/2 \leqslant z(x) \leqslant N - 1/2$, and $c_0, \dots, c_{N-1}$ are constants which depend on functions f and z.

The family of functionals E_ϵ Γ-converges as $\epsilon \to 0^+$ to a weak formulation of the piecewise constant Mumford–Shah functional with weighted length:

$$\mathcal{E}_w(K) = \sum_{j=1}^{M} w_j \mathcal{H}^1(K_j) + \lambda \sum_{i=0}^{N-1} \int_{\Omega_i} (c_i - f)^2 \, dx, \quad (4)$$

where $K = \cup_{j=1}^{M} K_j$ with $K_j \cap K_l = \emptyset$ for $j \neq l$, the set K and the phases Ω_i being defined as in (2), and the weights $w_1, \dots w_M$ are positive integers with $w_j \neq w_l$ for $j \neq l$. Here c_i is the mean value of f in Ω_i. We will argue that also the multiphase Chan–Vese model [59] yields a segmentation functional with weighted length like (4).

In the present paper, we extend the model (3) by considering functionals of the type:

$$E_\varphi^\epsilon(\mathbf{u}) = \int_\Omega \left[\epsilon^\alpha \varphi(|\nabla \mathbf{u}|) + \frac{W(\mathbf{u})}{\epsilon}\right] dx + \lambda \int_\Omega (\langle \mathbf{c}, \mathbf{u}\rangle - f)^2 \, dx, \quad (5)$$

where $\mathbf{u} : \Omega \to \mathbb{R}^N$ is a vector valued function, φ is a regularizing function whose properties will be described later, $\alpha \geqslant 0$ is a parameter, $\mathbf{c} \in \mathbb{R}^N$ is a constant vector, $W(\mathbf{u}) = \prod_{i=1}^{N} |\mathbf{u} - \mathbf{e}_i|^2$, and $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ is the standard basis of $\mathbb{R}^N$. The function W is called well-potential and the vectors $\mathbf{e}_1, \dots, \mathbf{e}_N$ at which W vanishes are called the wells of the potential W. We show that, for ϵ small enough, a multiphase segmentation is constituted by subsets of Ω (phases) where a minimizer $\mathbf{u}_\epsilon$ of E_φ^ϵ is close to a well $\mathbf{e}_i$, while the function $\mathbf{u}_\epsilon$ exhibits transitions across diffuse interfaces between any two phases. The diffuse interfaces have thickness of order ϵ and the sharpness of the transition depends on the regularizing function φ. When the regularizing function φ is non-convex, the functionals E_φ^ϵ extend to a vectorial setting the scalar model considered in [52,5] for edge-preserving image classification and restoration.

A motivation of the vectorial setting (5) is the following. Both Chan–Vese multiphase model and the model (3) do not correspond properly to the original Mumford–Shah model since the length term $\mathcal{H}^1(K)$ is replaced by a weighted length as in (4). This is an undesirable property since there are no geometric reasons to give a portion of the discontinuity set K a weight bigger than another portion, thus unbalancing the geometric term of the segmentation model. This drawback is common to many variational models considered in the literature.

The new segmentation functionals E_φ^ϵ permit us to overcome such a drawback while maintaining useful features of model (3). Indeed, following the theory of Γ-convergence, the functionals (5) give the true length term $\mathcal{H}^1(K)$ as $\epsilon \to 0^+$. We will show that the vectorial setting and the symmetry of the potential W with respect to the exchange of the wells $\mathbf{e}_i$ (not possible in a scalar setting) are crucial properties in order to achieve the true length. The main contributions of this paper are

- to propose models which give true length approximation via Γ-convergence, while giving sharp interfaces by choosing a suitable regularizing function φ;
- and to explore the differences and the similarities among these models both analytically and numerically.

Following the Mathematical definitions in Section 2, we carefully illustrate in Section 3 when multiphase segmentation models penalize a weighted length. In Section 4, various models are explored in the framework of Γ-convergence and using the well-potential model. Instead of going into the technical details of each convergence proof, we outline what is considered in the literature, and clearly mention what is not shown in the literature. We mainly focus on understanding the differences of the models. In Section 5, we review some of recent efficient numerical methods for multiphase segmentation. The numerical computation exhibits instabilities due to the non-convex shape of the proposed well-potential, yet we present the numerical results which are most true to the proposed model using a stochastic approach to validate these models.

2. Mathematical definitions

In the following, meas(A) denotes the two-dimensional Lebesgue measure of a set $A \subset \mathbb{R}^2$, and $\mathcal{H}^1(\partial A)$ denotes the one-dimensional Hausdorff measure of ∂A. We denote vectors in $\mathbb{R}^N$ and matrices in $\mathbb{R}^{2\times N}$ by means of bold symbols, if $\mathbf{u} \in \mathbb{R}^N$ we write $\mathbf{u} = (u^1, \dots, u^N)$. The scalar product of $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^N$ is denoted by $\langle \mathbf{a}, \mathbf{b}\rangle$. The norm $|\cdot|$ denotes the Euclidean norm both in $\mathbb{R}^N$ and in $\mathbb{R}^{2\times N}$. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary which represents the image domain. We will use standard notation for the Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$.

For any $N \in \mathbb{N}$, the space $BV(\Omega; \mathbb{R}^N)$ of functions of bounded variation mapping Ω to $\mathbb{R}^N$ is defined as the set of vector valued functions $\mathbf{u} \in L^1(\Omega; \mathbb{R}^N)$ such that $\int_\Omega |D\mathbf{u}| < +\infty$, where

$$\int_\Omega |D\mathbf{u}| = \sup_{\mathbf{G}\in\mathbf{\Psi}} \sum_{i=1}^{N} \int_\Omega u^i \mathrm{div}\, \mathbf{g}^i \, dx,$$

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