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A new continuous max-flow algorithm for multiphase image segmentation using super-level set functions

Jun Liu [a,*], Xue-cheng Tai [b], Shingyu Leung [c], Haiyang Huang [a]

[a] School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry Education, Beijing 100875, PR China
[b] Department of Mathematics, University of Bergen, Johaness Brunsgate 12, 5007 Bergen, Norway
[c] Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong

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ABSTRACT

We propose a graph cut based global minimization method for image segmentation by representing the segmentation label function with a series of nested binary super-level set functions. This representation enables us to use $K-1$ binary functions to partition any images into K phases. Both continuous and discretized formulations will be treated. For the discrete model, we propose a new graph cut algorithm which is faster than the existing graph cut methods to obtain the exact global solution. In the continuous case, we further improve the segmentation accuracy using a number of techniques that are unique to the continuous segmentation models. With the convex relaxation and the dual method, the related continuous dual model is convex and we can mathematically show that the global minimization can be achieved. The corresponding continuous max-flow algorithms are easy and stable. Experimental results show that our model is very competitive to some existing methods.

1. Introduction

Image segmentation is a fundamental but important task in computer vision and pattern recognition. It has received much attention by researchers during the past several decades. The objective of image segmentation is to partition an image into several parts according to some similarity measures such as intensity means, histograms, structure tensors and so on. There are many image segmentation methods proposed in the literature, in which the partial differential equation (PDE) based techniques and graph cut based approaches are two of the most popular image segmentation methods.

For PDE methods, the well known level set methods have been proven to be very flexible and quite efficient for image segmentation. The Mumford–Shah segmentation model [1] is an important approach to find a piecewise smooth approximation for a given image. However, the original Mumford–Shah functional is difficult to compute due to its weak mathematical properties such as discontinuity and non-convexity. By using a level set approximation, the discontinuity in the Mumford–Shah model can be easily handled and computed. To get a convex model, Cai, *etc.* [2] proposed a two-stage method for the Mumford–Shah segmentation model. Many methods of the fluid mechanics also can be applied to image segmentation, such as the phase-field method [3] and the Modica–Mortola phase transition method [4].

For the two-phase segmentation, the Chan–Vese model [5] is a very successful simplified version of the Mumford–Shah model. The Chan–Vese model is not convex. This explains why the numerical algorithm may sometimes get stuck at a local minimum close to the initial condition and produce undesirable segmentation results. Later, a binary level set method was proposed in [6] as a variant of the level set method. Meanwhile, the convex relaxation approach developed in [7] shows that one can get global minimizers for the piecewise constant Mumford–Shah functional with the binary approach [6] if we relax the binary constraint. The main idea of the convex relaxation is to relax the binary characteristic function into a continuous interval $[0,1]$ such that the non-convex original problem becomes convex. Solving such a relaxed convex problem can enable one to find a global minimizer, and then the global binary solution of the original problem can be obtained by a threshold process. Combining the convex relaxation and some recently developed TV (total variation) minimization techniques [8,9], Bresson, *etc.* have proposed some fast two-phase global minimization algorithms for image segmentation in [10,11].

For multi-phase segmentation, a generalization of the Chan–Vese model has been proposed in [12] to partition an image into n parts by using $\log_2 n$ level set functions. Similar to the two-phase case, the

* Corresponding author. Fax: +86 010 58808202.
E-mail address: jliu@bnu.edu.cn (J. Liu).

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model is non-convex and thus the global minimization cannot be guaranteed. Recently, a convex formulation of 4-phase Chan–Vese model has been proposed in [13,14] provided that the segmentation data term satisfies a convexity condition. Numerical tests show that this condition may be often satisfied in practice. In case this condition is violated, some "truncation" procedures have to be used.

Another multi-phase segmentation method is to use the label function or a PCLSM (piecewise constant level set method [15]) to represent different classes. By using a graph cut implementation, the PCLSM can be globally solved [16]. In the continuous case, functional lifting method [17] can be regarded as a convex formulation of PCLSM. As pointed out in [12,18], the TV of the label function or level set functions in PCLSM and multi-phase Chan–Vese model does not correspond exactly to the length term in Mumford–Shah model. The main drawback of these models is that some parts of the boundary are counted multiple times. Therefore pixels near some of the cluster boundaries will be misclassified (see e.g. [18]).

More recently, some continuous convex relaxation of the Potts model [19] have become popular. Bae, *etc.* proposed a smooth dual model of the Potts model in [20]. Pock, *etc.* [21] developed a tight convex relaxation framework for Potts model. Yuan, *etc.* [22,23] designed a max-flow approach to the Potts model. These continuous methods need to solve K unknown characteristic functions with a partition condition for K-phase clusters.

For the discrete partition problem, graph cut is a powerful tool to optimize the related energy. For example, the discrete Potts model restricted to 2-phase segmentation is computationally tractable by using some graph cut based min-cut/max-flow algorithms [24,25]. It is well known that the discrete Potts model is a NP-hard problem. Namely, if the number of segmentation classes is larger than two, there is no low-complexity algorithm which can find the exact global minimizer of the Potts model (see [26,27]). Instead of exactly solving the Potts model in a discrete setting, some algorithms for approximately minimizing the energy in Potts model have been proposed in [26], which are known as the popularly used alpha-expansion and alpha–beta swap algorithms.

Another approximation for the multi-phase Potts model is Ishikawa's graph cut method [28], in which the regularization term of the Potts model is modified such that it can be solved by a graph cut algorithm, c.f [28,16]. In 2006, Darbon, *etc.*[29] proposed another graph cut method for the multiphase segmentation. However, these graph-based methods generally suffer from metrication errors since the isotropic TV cannot be minimized by discrete max-flow algorithm. This difficulty could cause some zigzag edges in the clusters, which gives unnatural segmentation results. Recently, some continuous max-flow [30,22] algorithms have been developed by analyzing the primal min-cut and the dual max-flow problems with the Lagrangian multiplier method. These algorithms combine the advantages of both the continuous method and discrete model, and thus can provide impressive results.

This paper is devoted to propose a new graph cut method based on the multi-phase segmentation method, and the label function is represented by the binary super-level set. We will show that it is possible to minimize a modified piecewise constant Mumford–Shah segmentation model with the super-level set representation by solving the min-cut problem of a constructed graph.

The contributions of this paper include:

- We reveal the connections between the logical graph cut (discrete method) and the super-level set (continuous method) in the multiphase segmentation, and construct a graph which is related to the PCLSM using the super-level set expression. The proposed method is faster than the Ishikawa's graph cut method [28] due to some special structures of this graph.
- A continuous max-flow algorithm is proposed to further overcome two main drawbacks of the discrete graph cut method: one of the drawbacks of the discrete method is that the isotropic TV cannot be applied, but it can be employed in our method; the other one is that it is impossible to exactly penalize the length of the boundary for the multiphase segmentation in the discrete method, but the proposed method can do this. These two improvements ensure that one can get some better segmentation results from our algorithm. Compared to the Darbon's [29] discrete graph cut method, we proposed a continuous max-flow algorithm to overcome the drawbacks of the discrete method. Compared to some existing continuous methods, the proposed algorithm uses $K-1$ super-level set functions to partition K classes, which reduces the number of unknown variables, so providing a computationally very efficient algorithm. In addition, we use K dual variables to keep the regularization term in the model to be the exact length of the boundary in the continuous dual model, experimental results have shown that this can significantly improve the quality of the segmentation results.
- We give some mathematical analysis on the proposed algorithm, and show that the binary solutions of our algorithm can be obtained by a convex relaxation and a thresholding step, which corresponds to give a binary solution for the Potts model under a certain condition.

The rest of the paper is organized as follows: Section 2 gives some backgrounds on multi-phase segmentation methods; in Section 3, we introduce the proposed method, including the model, the algorithms and related analysis; Section 4 contains some experimental results; finally, some conclusions and discussions are presented in Section 5.

2. Related works

The generic problem of image segmentation is to partition an image domain Ω into K non-overlapping regions Ω_k such that $\Omega=\cup_{k=1}^{K}\Omega_k$. The well-known Potts model for image segmentation is to minimize the energy

$$\mathcal{E}^{Potts}\left(\{\Omega_k\}_{k=1}^{K}\right)=\sum_{k=1}^{K}\int_{\Omega_k}d^k(x)\mathrm{d}x+\mu\sum_{k=1}^{K}|\partial\Omega_k|, \tag{1}$$

such that $\cup_{k=1}^{K}\Omega_k=\Omega$ and $\Omega_i\cap\Omega_j=\emptyset$ if $i\neq j$, where $|\partial\Omega_k|$ stands for the perimeter of the boundary of Ω_k and $\mu>0$ is a parameter. Here the first term is the data term, and each d^k should depend on the input image I. For example, $d^k(x)=|I(x)-c^k|^\lambda,\ \lambda=1,2$ represents that the pixels are classified in terms of the intensity means $\{c^k\}_{k=1}^{K}$. The second term, namely the regularization term, measures the sum of the perimeters of the sets $\Omega_k, k=1,\dots,K$. When $\lambda=2$ and $\{c^k\}_{k=1}^{K}$ are unknown, (1) coincides with the energy of the piecewise constant Mumford–Shah model [1]:

$$\mathcal{E}^{MS}\left(\{\Omega_k\}_{k=1}^{K},\{c^k\}_{k=1}^{K}\right)=\sum_{k=1}^{K}\int_{\Omega_k}|I(x)-c^k|^2\mathrm{d}x+\mu\sum_{k=1}^{K}|\partial\Omega_k|. \tag{2}$$

By introducing a vector-valued characteristic function $\boldsymbol{\psi}(x)=(\psi^1(x),\dots,\psi^K(x))$ with component functions

$$\psi^k(x)=\begin{cases}1, & x\in\Omega_k,\\ 0, & x\notin\Omega_k,\end{cases}$$

the Potts model (1) can be reformulated as

$$\mathcal{E}^{Potts}(\boldsymbol{\psi})=\sum_{k=1}^{K}\int_{\Omega}d^k(x)\psi^k(x)\mathrm{d}x+\frac{\mu}{2}\sum_{k=1}^{K}\int_{\Omega}|\nabla\psi^k(x)|\mathrm{d}x, \tag{3}$$