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A modified convex variational model for multiplicative noise removal ☆,☆☆

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ABSTRACT

In this paper, a convex variational model for multiplicative noise removal is studied. Accelerating primal–dual method and proximal linearized alternating direction method are also discussed. An improved primal–dual method is proposed. Algorithms above produce more desired results than primal–dual algorithm when we solve the convex variational model. Inspired by the statistical property of the Gamma multiplicative noise and I-divergence, a modified convex variational model is proposed, for which the uniqueness of solution is also provided. Moreover, the property of the solution is presented. Without inner iterations, primal–dual method is efficient to the modified model, and running time can be reduced dramatically also with good restoration. When we set parameter α to 0, the convex variational model we proposed turns into the model in Steidl and Teuber (2010). By altering α, our model can be used for different noise level.

1. Introduction

Image denoising is an important problem in signal and image processing and has being widely studied in the applied mathematics community. Most of literatures deal with additive noise model: an original image u, which is degraded by some additive noise η, and the corrupted image f. The problem is to recover u from f. A mathematical description of such degradation process as follows, assume that $f:\Omega\rightarrow R$, is a real function, where Ω is a connected bounded open subsets of R^2 with Lipschitz boundary, and f is generated from a general model $f=u+\eta$. The recovery of u from f is an ill-posed inverse problem. Many methods and various effective algorithms have been proposed, and we are interested in the variational approach here. One of the most successful and popular techniques for approximating the solution of this problem is proposed by Rudin–Osher–Fatemi (cf. [28]), i.e., ROF model, which is defind as follows:

$$u=\arg\min|u|_{BV}+\lambda\|u-f\|_{L_2}^2,$$

for $\lambda>0$, where $BV(\Omega)$ denotes the space of functions with bounded variation on Ω, see the following for more details, and $|\cdot|$ denotes the BV seminorm, formally given by

$$|u|_{BV}=\int|\bigtriangledown u|,$$

which is also marked as total variation (TV) of u. An overview of image restoration based on variational regularization is presented in [16]. In this paper, we focus on restoring images which are corrupted by multiplicative noise η, i.e., restoring the original image u from $f=u\eta$, where η is assumed to follow the Gamma distribution with mean 1. For $x\geqslant 0$

$$P_\eta(x,\theta,K)=\frac{1}{\theta^K\Gamma(K)}x^{K-1}e^{-\frac{x}{\theta}}.$$

Usually we assume that $f>0$, and unlike additive noise removal problems, the noisy image is the multiplication of noise and the original image, so almost all information of the original image may disappear in the observed image. Hence, multiplicative noise removal is totally different from additive noise removal, and it commonly occurs in active imaging systems, like synthetic aperture radar (SAR), ultrasound imaging, microscope images, etc.

1.1. Multiplicative noise removal

The multiplicative noise has not yet been researched thoroughly, especially with variational method. As far as we know,

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the first variational approach devoted to multiplicative noise is the one by Rudin et al. [27], a constrained optimization problem which is referred to as RLO model:

$$\inf_{u \in S(\Omega)} \int_\Omega |Du| \tag{1}$$

subject to

$$\int_\Omega \frac{f}{u} dx = 1,$$

$$\int_\Omega \left(\frac{f}{u} - 1\right)^2 dx = \theta^2,$$

where θ^2 denotes the variance of $\eta, S(\Omega) = \{v \in BV(\Omega) : v > 0\}, BV(\Omega)$ is the subspace of functions space $u \in L^1(\Omega)$ such that the following is finite:

$$J(u) = sup\left\{\int_\Omega u(x) div\xi(x)dx, \xi \in C_0^\infty(\Omega, R^2), ||\xi||_{L^\infty(\Omega,R^2)} \le 1\right\}, \tag{2}$$

where $C_0^\infty(\Omega)$ is the set of smooth functions on Ω that vanishes on the boundary $\partial\Omega$. $BV(\Omega)$ equipped with the norm $||u||_{BV} = ||u||_{L^1} + J(u)$ is a Banach space. If $u \in BV(\Omega)$, the distributional derivative Du is a bounded Radon measure, and (2) corresponds to the total variation, i.e., $J(u) = \int_\Omega Du$. For $\Omega \subset R^2$, if $1 \leqslant p \leqslant 2$, then $BV(\Omega) \hookrightarrow L^p(\Omega)$, further more, for $1 \leqslant p < 2$, this embedding is compact. For further details on $BV(\Omega)$, we refer readers to [2]. Considering that it can smooth the noise in the isotropic regions of noisy image and can also protect some important details from over-smoothing, the total variation (TV) of u is selected as the objective function. With the maximum a posteriori (MAP) estimator, Aubert and Aujol proposed a variational model as follows, and generally, it is referred to as AA model [3],

$$\inf_{u \in S(\Omega)} \int_\Omega \left(\log u + \frac{f}{u}\right) dx + \lambda \int_\Omega |Du|. \tag{3}$$

The first term is the fidelity term, and the second term is a regularization term. Parameter λ is a trade-off between a good fit of f and a smoothness requirement. In order to restore more texture details of the denoised image, spatially varying regularization parameter is employed in AA model [24]. With a nonconvex fidelity, the computed solutions by some optimization methods are not necessary to a global optimal solution, and also rely on the initial estimation, and thus, the restored quality is affected. In order to obtain a convex model, a logarithm transform on both sides of equation $f = u\eta$ is taken in [29], and the multiplicative problem is converted into an additive one which we denote it as log-model. A rather general formulation for the multiplicative noise is also presented in [29], including [2,11,27]. Then log-model is modified in [21] by introducing an auxiliary variable $w = \log u$ and a quadratic term, and a simpler alternating minimization algorithm is performed. Also, the convergence of this algorithm is described. It is remarkable that the model in [5,21,29] are convex in the logarithm domain, but are not convex in the original image domain. In [30], the I-divergence as the data fidelity term with TV regularization or the nonlocal means to remove the multiplicative Gamma noise is considered, and in the continuous setting, the relationship between the log-model and the I-divergence-TV multiplicative noise model is built. With rewriting the blur and multiplicative noise equation such that both the image variable and the noise variable are decoupled, a new convex optimization model is proposed in [36]. In [13], the unconvexity of AA model is improved by a quadratic term which is based on statistic properties of the gamma noise, and the uniqueness of the solution is demonstrated under a mild condition. Since the model is improved to convex, algorithms for solving convex optimization problems can be used. For instance, the alternating direction method of multipliers (ADMM) in [18], its variant, the split-Bregman algorithm in [19], proximal linearized alternating direction (PLAD) method in [31], Chambolle's semi-implicit gradient decent method [7], and the primal–dual hybrid gradient algorithm [35,37], etc. Primal–dual method which is proposed in [8,15,26] is applied in [13]. For multiplicative noise, the filter-based methods are also very popular. The median filter has also been examined for multiplicative noise reduction in [10,25]. Nonlocal means method has been studied in [6], and nonlocal operators is describes in [22]. Owing to the similarity of image patches, the nonlocal regularization can preserve image edges and textures better than the classical regularization. Some convex model with the nonlocal total variation for multiplicative noise removal has been discussed. In [12], the log-model with the nonlocal regularization term has been demonstrated. Also in [30], nonlocal means as regularizer is described.

1.2. Some properties of multiplicative noise

Considering a random variable $Y = \frac{1}{\sqrt{\eta}}$, where η follows Gamma distribution. Some statistic properties has been discussed in previous articles, we make a simple depiction here.

Proposition 1.1. *Suppose that the random variable η follows a Gamma distribution with mean 1. Set $Y = \frac{1}{\sqrt{\eta}}$ then*

$$\lim_{K\to\infty} E((Y-1)^2) = 0.$$

Proposition 1.2. *Suppose that the random variable η follows a Gamma distribution with mean 1. Set $Y = \frac{1}{\sqrt{\eta}}$. Then we have the following:*

(1) $\int_0^{+\infty} p_Y(y) \log p_Y(y) dy = \log 2 - \log(\sqrt{K}\Gamma(K)) + \frac{2K+1}{2}\Psi(K) - K$, where $\Psi(K) := \frac{d\log\Gamma(K)}{dK}$ is the digamma function (see [1]);
(2) $\int_0^{+\infty} p_Y(y) \log p_{N(\mu_K,\sigma_K^2)}(y) dy = -\frac{1}{2}\log(2\pi e\sigma_K^2)$, where $p_{N(\mu_K,\sigma_K^2)}(y)$ denotes the probability density function(*PDF*) of the Gaussian distribution $N(\mu_K, \sigma_K^2)$;
(3) $\lim_{K\to+\infty} D_{KL}(Y\|N(\mu_K, \sigma_K^2)) = 0$.

The proof were discussed in [13]. But as we can see there, the value of $E((Y-1)^2)$ is small only when K is relatively large, and the Fig. 1 in [13] also demonstrates that when K is sufficient large, the *KL* divergence of Y with the Gaussian distribution $N(\mu_K, \sigma_K^2)$ tends to 0, where μ_K and σ_K^2 are the mean and variance of Y, respectively. When $K = 6$, see Fig. 1 of [13], it does not fit very well.

By introducing a quadratic penalty term and a parameter α into the AA model (3), it leads to a convex variatioal model discussed by Dong and Zeng [13]:

$$\inf_{u \in \bar{S}(\Omega)} \int_\Omega \left(\log u + \frac{f}{u}\right) dx + \alpha \int_\Omega \left(\sqrt{\frac{u}{f}} - 1\right)^2 dx + \lambda \int_\Omega |Du|. \tag{4}$$

The penalty parameter $\alpha > 0$, besides, $\bar{S}(\Omega) := \{v \in BV(\Omega) : v \geqslant 0\}$, which is closed and convex, and $\log 0 = -\infty, \frac{1}{0} = +\infty$ is defined. The existence and uniqueness of solution is presented with a proper selection of α.

1.3. The contribution

Although model (4) is convex with a proper α, and many algorithms for convex optimization can be used, but the fidelity