J. Vis. Commun. Image R. 24 (2013) 75–80

Contents lists available at SciVerse ScienceDirect

J. Vis. Commun. Image R.

journal homepage: www.elsevier.com/locate/jvci

A bivariate rational interpolation based on scattered data on parallel lines

Qinghua Sun [a], Fangxun Bao [a,*], Yunfeng Zhang [b], Qi Duan [a]

[a] School of Mathematics, Shandong University, Jinan 250100, China
[b] School of Computer Science and Technology, Shandong Economic University, Jinan 250014, China

ARTICLE INFO

Article history:
Received 13 March 2011
Accepted 20 November 2012
Available online 29 November 2012

Keywords:
Rational spline
Scattered data
Triangulation
Bivariate interpolation
Computer-aided geometric design
Shape control

ABSTRACT

In many practical problems, such as geological exploration, forging technology and medical imaging, among others, it has been detected that the scattered data are usually arranged in parallel lines. In this paper, a new approach to construct a bivariate rational interpolation over triangulation is presented, based on scattered data in parallel lines. The main advantage of this method comparing with the present interpolation methods have two points: (1) the interpolation function is carried out by a simple and explicit mathematical representation through the parameter α; (2) the shape of the interpolating surface can be modified by using the parameter for the unchanged interpolating data. Moreover, a local shape control method is employed to control the shape of surfaces. In the special case, the method of "Barycenter Value Control" is studied, and numerical examples are presented to show the performance of the method.

1. Introduction

The construction method of curves and surfaces and their mathematical description is a key issue in Computer-Aided Geometric Design (CAGD). There are many ways to tackle this problem [4,6,7,10,19,24–26,29,30], for example, the polynomial spline method, the NURBS method and the Bézier method. These methods are applied widely in the shape design of industrial products. Specifically, most of the polynomial spline methods are the interpolating methods. However, one of the disadvantages of the polynomial spline method is that the local shape can not be modified for the interpolating surfaces while interpolating data is unchanged. The Non Uniform Rational B Splines (NURBS) and Bézier methods are the so-called "no-interpolating type" methods; this means that the constructed curve and surface do not fit with the given data, so the given points play the role of the control points. Thus, in order to construct the interpolating functions required for CAGD, the following conditions must be satisfied: (a) the interpolating functions achieve simple and explicit representations, so that these representations can be conveniently used for both practical application and theoretical analysis; (b) the parameters of constructed curves and surfaces can be modified without changing the given data.

In recent years, the study of univariate rational spline interpolation with parameters has received attention in the literature, and many results have been established [1,2,9,11,12,20,21,27]. Motivated by the univariate rational spline interpolation, the bivariate rational spline, which has a simple and explicit mathematical representation with parameters, has been studied. Since the parameters in the interpolation function are selective according to the control constrains, the constrained control of the shape becomes possible. In [14–16], several bivariate spline interpolations have been constructed over rectangular mesh, and properties have been also derived, such as, the sufficient conditions of "down-constrained" and "up-constrained" for the shape control of interpolating surfaces are obtained in [14], the matrix expression and the bounded property of interpolation function and the properties of the integral weight coefficients are derived in [15], the properties of the integral weight coefficients and the stability of interpolation are discussed in [16]. In [31], convexity control of a bivariate rational interpolating spline surfaces is studied. In [23], the preserving positivity of a rational bicubic spline interpolation with parameters over a rectangular grid is discussed. But, in many practical problems, the rectangular mesh is very difficult to be calculated, because only the scattered data can be obtained to achieve an interpolation. Thus, it is necessary to construct the bivariate interpolation function over the triangulation lattice. There are many publications contributing to the bivariate spline interpolation over triangulation. For example, in [6,7,18], the structure of bivariate spline spaces is investigated, and the important applications of Bernstein–Bézier techniques in CAGD are discussed; in [8], a characterization of smoothness of polynomial pieces on adjacent triangles, without using Bernstein-Bézier techniques, is proved; in [5], the approximation order from the space $S := \prod_{k,\Delta}^{\rho}$ of piecewise polynomial functions is studied; in [3], an adaptive quasi-interpolating quartic spline based on a uniform quasi-interpolating scheme is constructed over a regular triangular mesh, and the adaption of

* Corresponding author.
E-mail address: fxbao@sdu.edu.cn (F. Bao).

1047-3203/$ - see front matter © 2012 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jvcir.2012.11.003

the scheme to surfaces of varying geometric complexity can be locally defined, etc. The all above bivariate spline interpolations over triangulation are in fact polynomial interpolations. In [30], the structure of bivariate rational spline spaces on arbitrary triangulation is investigated by using the methods of smoothing cofactor, and some commonly used methods for studying multivariate spline functions, such as B-spline method, B-net method and the integral methods, etc., are clearly explained. Here we are concerned with bivariate rational spline interpolations with a simple and explicit mathematical representation, which can be modified by using new parameters. In many practical problems, such as geological exploration, forging technology and medical imaging, among others, it has been detected that the scattered data are usually arranged in parallel lines, as shown in Fig. 1, where the triangulation is simply obtained using a few number of control parameters. In this paper, the construction problem of the bivariate spline interpolation over a triangulation mesh is considered. To solve the problem, a new approach is proposed by using a constructed interpolation function comprising a simple and explicit mathematical representation with the new parameter α. This parameter can be modified by using the parameter being achieved for the unchanged interpolating data. Also, a local shape control method of interpolating surface is developed.

This paper is arranged as follows. In Section 2, a new bivariate rational spline interpolation with parameter is constructed over triangulation, using the scattered data in parallel lines. Section 3 is about some properties of the interpolation function, including the properties of the basis function and the bounded property. Section 4 deals with the error estimates of the interpolation function. In Section 5, the shape control method of the interpolating surface is given, in the special case, the "Barycenter value control" is studied, and various numerical examples are presented to show the performance of the method.

2. Definition of interpolation function

Let $\{(x_i, y_i, f_i, d_i),\ i = 1, 2, \ldots, n\}$ be the given scattered data arranged in parallel lines: $e_1, e_2, \ldots, e_m$, where $f_i = f(x_i, y_i)$, and $d_i = \frac{\partial f(x_i, y_i)}{\partial x}$ (see Fig. 1).

For a triangular domain $T_1 = \triangle V_1V_2V_3$,with vertices $\{V_i = (x_i, y_i),\ i = 1, 2, 3\}$ and $y_1 = y_2$, let $\gamma_{11} = \angle V_3V_1V_2$ be the angle between lines V_3V_1 and V_1V_2, and γ_{12} be the angle between line V_2V_3 and the extension line of V_1V_2 (see Fig. 2).

Denoting $h = x_2 - x_1, l = y_3 - y_1$. For any point Q in the line V_1V_2, let $\beta = \angle V_3QV_2$ be the angle between lines V_3Q and QV_2, thus, $V_1Q = x_3 - x_1 - l\cot\beta$, and for any point $V(x, y)$ in the line V_3Q, $\cot\beta = \frac{x - x_3}{y - y_3}$. Let $\theta = \frac{x_3 - x_1 - l\cot\beta}{h}$, and $\eta = \frac{y - y_1}{l}$. A rational cubic function is defined over the interval $[x_1, x_2]$ as [12]

$$p(x) = \frac{(1-\theta)^3\alpha f_1 + \theta(1-\theta)^2 V + \theta^2(1-\theta)W + \theta^3 f_2}{(1-\theta)\alpha + \theta}, \tag{1}$$

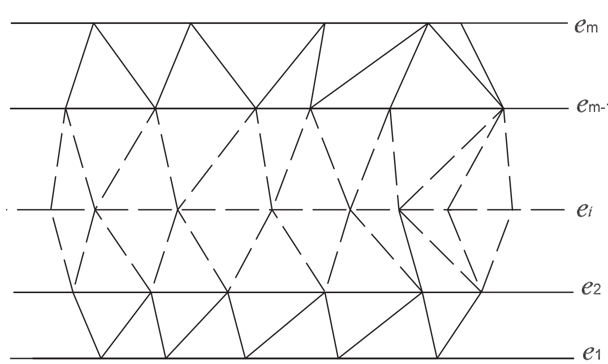

Fig. 1. Triangulation of interpolating region Ω.

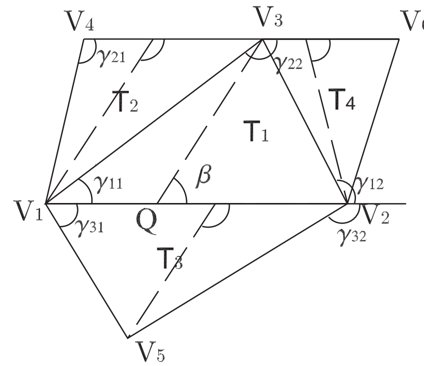

Fig. 2. An element Ω^* of subdivision.

where

$$V = (2\alpha + 1)f_1 + \alpha h d_1,$$
$$W = (\alpha + 2)f_2 - h d_2,$$

with $\alpha > 0$. Obviously, the interpolation function $p(x)$ on $[x_1, x_2]$ is unique for the given data (x_i, f_i, d_i), $i = 1, 2$ and the parameter α, and which satisfies

$$p(x_i) = f_i, \quad p'(x_i) = d_i; \quad i = 1, 2.$$

Using the x-direction interpolation function $p(x)$, we define the bivariate rational interpolation function $P(x, y)$ on the triangular domain T_1 as follows:

$$P_{T_1}(x, y) = (1-\eta)p(x) + \eta f_3. \tag{2}$$

It is called the bivariate rational interpolator on the triangular domain T_1.

Defining $P_{T_1}(x_3, y_3) = f_3$, then

$$\lim_{y \to y_3^-} P_{T_1}(x_3, y) = f_3.$$

Thus, the interpolation function satisfies

$$P_{T_1}(x_i, y_i) = f_i; \quad i = 1, 2, 3.$$

Let T_2, T_3 and T_4 be the triangular domains which have common edges V_1V_3, V_1V_2 and V_2V_3 with T_1 respectively. Denoting $\Omega^* = T_1 \bigcup T_2 \bigcup T_3 \bigcup T_4$, then the subregion Ω^* of interpolating region Ω is called an element of subdivision (see Fig. 2).

For the continuity of the interpolation function defined by Eq. (2) in the whole interpolating region Ω, it is only necessary to consider the continuity in an element Ω^* of subdivision. Then by the symmetry of T_2 and T_4, it is only necessary to show the continuity of the interpolation function in lines V_1V_2 and V_1V_3, respectively.

Similar to Eq. (2), we can define the bivariate rational interpolation functions over T_2 and T_3.

$$P_{T_2}(x, y) = (1-\eta)f_1 + \eta p^*(x),$$

where

$$p^*(x) = \frac{1}{(1-\theta)\alpha^* + \theta}\Big[(1-\theta)^3\alpha^* f_4 + \theta(1-\theta)^2((2\alpha^*+1)f_4 + \alpha^* h^* d_4) + \theta^2(1-\theta)((2+\alpha^*)f_3 - h^*, d_3) + \theta^3 f_3\Big],$$

with

$$\theta = \frac{x_1 - x_4 - l\cot\beta}{h^*}, \quad \cot\beta = \frac{x - x_1}{y_1 - y}, \quad h^* = x_3 - x_4,$$
$$\eta = \frac{y - y_1}{y_3 - y_1}, \quad \alpha^* > 0.$$

$$P_{T_3}(x, y) = (1-\eta)f_5 + \eta p(x),$$