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Lagrangian multipliers and split Bregman methods for minimization problems constrained on $\mathcal{S}^{n-1}$

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ABSTRACT

The numerical methods of total variation (TV) model for image denoising, especially Rudin–Osher–Fatemi (ROF) model, is widely studied in the literature. However, the $\mathcal{S}^{n-1}$ constrained counterpart is less addressed. The classical gradient descent method for the constrained problem is limited in two aspects: one is the small time step size to ensure stability; the other is that the data must be projected onto $\mathcal{S}^{n-1}$ during evolution since the unit norm constraint is poorly satisfied. In order to avoid these drawbacks, in this paper, we propose two alternative numerical methods based on the Lagrangian multipliers and split Bregman methods. Both algorithms are efficient and easy to implement. A number of experiments demonstrate that the proposed algorithms are quite effective in denoising of data constrained on S^1 or S^2, including general direction data diffusion and chromaticity denoising.

1. Introduction

Variational denoising methods have become popular in recent years, for instance, the well known Rudin–Osher–Fatemi (ROF) model [26] and its various extensions [11,16,19]. The scalar ROF model for gray-scale image is:

$$\min_u \int_\Omega |\nabla u| dx + \frac{\lambda}{2}\int_\Omega (f-u)^2 dx, \tag{1}$$

where f is the observed noisy image and λ is a positive balance parameter. Here, the first term is called total variation (TV) which is widely used as a regularization term in variational image processing approaches [1]. In the past decades, a large amount of fast numerical schemes instead of the gradient descent methods are proposed to handle the TV based minimization models. For instance, the Chambolle's fast dual method [7], the alternating split Bregman method [17], the operator splitting method [12,20,22], the alternating direction method of multipliers (ADMM) [15,24], the primal–dual method [8,13] and some other methods [2,3,25,23,33].

Let us now write down the n-dimensional ROF model constrained on $\mathcal{S}^{n-1}$. Assume $\Omega \subset \mathbb{R}^2$ is an open bounded domain, and $\mathbf{f} : \Omega \to \mathcal{S}^{n-1} \subset \mathbb{R}^n$ is the observed noisy data and $\mathbf{u} : \Omega \to \mathbb{R}^n$ is a vectorial function. The general problem can be formulated as:

$$\min_{\mathbf{u}} E(\mathbf{u}) = \int_\Omega \|\nabla \mathbf{u}\| dx + \frac{\lambda}{2}\int_\Omega |\mathbf{u}-\mathbf{f}|^2 dx \quad s.t.\ |\mathbf{u}| = 1, \tag{2}$$

where

$$\nabla \mathbf{u} = \begin{pmatrix} \nabla u_1 \\ \nabla u_2 \\ \vdots \\ \nabla u_n \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \\ \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} \end{pmatrix},$$

with $x = (x_1, x_2)$ denotes the coordinates in image domain Ω,

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

and

$$\|\nabla \mathbf{u}\| = \sqrt{|\nabla u_1|^2 + |\nabla u_2|^2 + \cdots + |\nabla u_n|^2}.$$

In fact, $\int_\Omega \|\nabla \mathbf{u}\| dx$ is a generalization of color TV [4]. Remark that the problem is nonconvex since the constraint $|\mathbf{u}| = 1$ is not convex.

The above model can be used for direction data diffusion where the direction data has unit norm. An example in image processing field is chromaticity denoising. Although most of the variational denoising models use Right-Green-Blur (RGB) color model, there are some methods use other color models especially Chromaticity-Brightness (CB) color model. In the CB color model, the chromaticity component $\mathbf{u}$ and the brightness component B can be calculated as follows:

$$B = |\mathbf{u}| := \sqrt{u_1^2 + u_2^2 + u_3^2}, \quad \mathbf{C} = \frac{\mathbf{u}}{|\mathbf{u}|}.$$

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Note that the chromaticity $\mathbf{C}$ is a vector lives on the unit sphere in $\mathbb{R}^3 : \mathcal{S}^2 = \{\xi \in \mathbb{R}^3 : |\xi| = 1\}$. Therefore chromaticity is belonging to non-flat image feature differs from other features defined in Euclidean space. The CB model is known to be closer to human perception which is widely used in color image representation and modeling. In [9], it is shown that using CB color model gives better color control and detail recovery for color image denoising compared with other color models.

In literatures, some methods are introduced to handle the minimization problems on $\mathcal{S}^{n-1}$. Tang et al. in [29,30] proposed to denoise chromaticity or general direction data via p-harmonic maps in liquid crystals. The classical gradient descent method is used to solve the corresponding Euler–Lagrange equation which is limited by small time step and converges slowly. Recall that the gradient descent method for problem (2) is the flow [9]:

$$\frac{\partial \mathbf{u}}{\partial t} = \operatorname{div}\left(\frac{\nabla \mathbf{u}}{\|\nabla \mathbf{u}\|}\right) - \mathbf{u}\|\nabla \mathbf{u}\| + \lambda(\mathbf{f} - \langle \mathbf{f}, \mathbf{u}\rangle \mathbf{u}). \tag{3}$$

More generally, Tschumperl and Deriche in [31] studied the orthonormal vector sets diffusion problem by ϕ-function regularization [1] and the related negative gradient flow. In [32], Vese and Osher changed the constrained p-harmonic problem:

$$\min_{|\mathbf{u}|=1} \int_\Omega \|\nabla \mathbf{u}\|^p dx,$$

as an unconstrained one:

$$\min_V \int_\Omega \left\|\nabla\left(\frac{V}{|V|}\right)\right\|^p dx.$$

Numerically, the gradient descent method with implicit scheme is applied to evolve V based on polar coordinates. With similar idea, Cecil et al. in [6] proposed numerical methods for minimization problems constrained on $\mathcal{S}^1$ and $\mathcal{S}^2$ by technique based on the angle formulation, and numerically gradient descent method is used. In [10], Chan and Shen used vectorial ROF model to denoise non-flat data. Numerically, they developed fixed-point iteration. Bresson and Chan in [5] extended Chambolle's dual algorithm to vectorial ROF model, meanwhile, they generalize the algorithm to denoise the chromaticity component in color image. In [18], Haehnle and Prohl proposed discrete finite element based algorithms to approximate the L^2 gradient flow of the Mumford–Shah–Euler functional for unit vector fields and applied the algorithms in color image inpainting. In [34], Goldfarb et al. proposed new gradient descent algorithms for the p-harmonic flow problem on spheres, which searches the step along a curve that lies on the sphere and can preserve the pointwise sphere constraints. The method is generalized by Wenand Yin in [35] to handle the general orthogonal constraints.

In this paper, we consider two alternative numerical algorithms to solve problem (2) constrained on $\mathbf{S}^{n-1}$. Our main idea is to split the original problem into easier subproblems by introducing auxiliary variables. In Algorithm 1, we first use the standard Lagrangian method to handle the pointwise unit norm constraint, and then relax the energy by adding an auxiliary variable. In Algorithm 2, we first derive an equivalent problem with two auxiliary variables and three constraints, and then use the split Bregman method to handle the constraints. In both methods, all the involved subproblems are easy to solve.

The outline of this paper is as follows. In Section 2, we propose our Algorithm 1 based on Lagrangian multipliers method. In Section 3, we develop our Algorithm 2 based on the so called split Bregman method. The numerical results including direction data diffusion on S^1 and chromaticity denoising on S^2 are reported in Section 4. Finally, we conclude the paper in Section 5.

2. Algorithm 1 – Lagrangian multipliers method

In this section, we propose the Algorithm 1 to solve problem (2). Since the pointwise constraint $\mathbf{u}(x) = 1$ is equivalent to $|\mathbf{u}(x)|^2 - 1 = 0$, by using Lagrange multipliers method on the constraints we get an equivalent unconstrained problem:

$$\min_{\mathbf{u},\mu}\left\{\begin{array}{l} E_1(\mathbf{u},\mu) = \int_\Omega \|\nabla \mathbf{u}\| dx + \frac{\lambda}{2}\int_\Omega |\mathbf{u}-\mathbf{f}|^2 dx \\ + \frac{1}{2}\int_\Omega \mu(x)(|\mathbf{u}(x)|^2 - 1)dx \end{array}\right\}, \tag{4}$$

where $\mu(x)$ is the Lagrange multiplier at point $x \in \Omega$. The problem is not easy to solve since TV term is nonsmooth. In order to find an efficient algorithm, we consider an approximate problem by adding new variables such that the new problem is easy to solve. We add a new variable $\mathbf{v}$ to approximate $\mathbf{u}$ and obtain an approximate problem:

$$\min_{\mathbf{u},\mathbf{v},\mu}\left\{\begin{array}{l} E_2(\mathbf{u},\mathbf{v},\mu) = \int_\Omega \|\nabla \mathbf{v}\| dx + \frac{1}{2\theta}\int_\Omega |\mathbf{v}-\mathbf{u}|^2 dx \\ + \frac{\lambda}{2}\int_\Omega |\mathbf{u}-\mathbf{f}|^2 dx + \frac{1}{2}\int_\Omega \mu(x)(|\mathbf{u}|^2 - 1)dx \end{array}\right\}, \tag{5}$$

where θ is small enough to ensure that $\mathbf{u}$ almost equals $\mathbf{v}$. In the following subsections, we will derive the formulas for updating $\mathbf{u}$, μ and $\mathbf{v}$ in problem (5), respectively with alternating minimization method.

2.1. Solving u

Fixing μ and $\mathbf{u}$, the subproblem for $\mathbf{u}$ is:

$$\min_{\mathbf{u}}\left\{\begin{array}{l} \frac{1}{2\theta}\int_\Omega |\mathbf{v}-\mathbf{u}|^2 dx \\ + \frac{\lambda}{2}\int_\Omega |\mathbf{u}-\mathbf{f}|^2 dx + \frac{1}{2}\int_\Omega \mu(x)(|\mathbf{u}|^2 - 1)dx \end{array}\right\}. \tag{6}$$

The corresponding Euler–Lagrange equation about $\mathbf{u}$ is:

$$\frac{1}{\theta}(\mathbf{u}-\mathbf{v}) + \lambda(\mathbf{u}-\mathbf{f}) + \mu\mathbf{u} = 0. \tag{7}$$

Then we derive the closed-form solution of $\mathbf{u}$:

$$\mathbf{u} = \frac{\mathbf{v} + \lambda\theta\mathbf{f}}{1 + \lambda\theta + \mu\theta}. \tag{8}$$

2.2. Solving the Lagrange multipliers μ

Taking derivative of E_2 with respect to μ and setting it to zero, we get:

$$|\mathbf{u}|^2 = \langle \mathbf{u}, \mathbf{u}\rangle = 1, \tag{9}$$

for each $x \in \Omega$, where $\langle\cdot\rangle$ denotes the inner product in $\mathbb{R}^3$. Taking the inner product of (7) with $\mathbf{u}$ and using (9), we obtain the closed-form solution of μ:

$$\mu = \frac{1}{\theta}\langle \mathbf{u}, \mathbf{v}\rangle + \lambda\langle \mathbf{u}, \mathbf{f}\rangle - \frac{1}{\theta} - \lambda. \tag{10}$$

Remark that the above formula (10) was also derived in [21] and successfully used in colorization problems.

2.3. Solving auxiliary variable v

Fixing $\mathbf{u}$, the subproblem for $\mathbf{v}$ is:

$$\min_{\mathbf{v}} \int_\Omega \|\nabla \mathbf{v}\| dx + \frac{1}{2\theta}\int_\Omega |\mathbf{v}-\mathbf{u}|^2 dx, \tag{11}$$

which is a standard vectorial ROF model. Recall that many fast numerical algorithm have been designed to solve the scalar ROF model, see Section 1. These fast algorithms can be directly used to solve vectorial ROF model when a channel by channel TV is used. That is because in every channel the problem becomes a scalar