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Complete lattice learning for multivariate mathematical morphology ☆

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ABSTRACT

The generalization of mathematical morphology to multivariate vector spaces is addressed in this paper. The proposed approach is fully unsupervised and consists in learning a complete lattice from an image as a nonlinear bijective mapping, interpreted in the form of a learned rank transformation together with an ordering of vectors. This unsupervised ordering of vectors relies on three steps: dictionary learning, manifold learning and out of sample extension. In addition to providing an efficient way to construct a vectorial ordering, the proposed approach can become a supervised ordering by the integration of pairwise constraints. The performance of the approach is illustrated with color image processing examples.

1. Introduction

Mathematical Morphology (MM) is a nonlinear approach to image processing based on the application of lattice theory to spatial structures in images. The construction of morphological operators requires the definition of a complete lattice structure, i.e., an ordering between the elements to be processed. With the acceptance of complete lattice theory, it is possible to define morphological operators for any type of multivariate image data once a proper ordering is established [1]. However, if MM is well defined for binary and gray scale images, there exists no general admitted extension that permits to perform morphological operations on multivariate data since there is no natural ordering on vectors. Indeed, it is difficult to define an effective ranking of vectors in arbitrary vector spaces as well as determining the infimum and the supremum between vectors of more than one dimension. Therefore, the extension of mathematical morphology to multivariate images is a very active field. We refer the reader to [2,3,1] for a comprehensive review of vector morphology. Several recent approaches have been proposed in literature for e.g., color and hyperspectral images [4–10].

This paper introduces a systematic approach towards the construction of complete lattices for any kind of multivariate data. Following recent approaches [8,9], we propose to learn, in an unsupervised manner, the construction of a complete lattice from the values of an image. To do so, we rely on the theoretical framework of h-orderings [11], suitable for the definition of complete lattices. This framework requires the definition of a bijective mapping operator, and we propose to define the latter by nonlinear manifold learning directly from the set of vectors of the image under consideration. This problem being practically too computationally demanding, we propose a three-step strategy towards the construction of the mapping.

The paper is organized as follows. In Section 2, we explain in details the difficulty of the definition of complete lattices in vectors spaces. The properties of orderings and the associated taxonomy [12] are recalled, and the concept of complete lattices is introduced as well as how mathematical morphology operators operate on the latter. We detail what orderings are relevant for morphological processing of multivariate vectors and why the framework of h-ordering is a very appealing approach. Then we show different interpretations of this framework and interpret it as a rank transform. Section 3 presents our approach for the learning of a complete lattice. First, a reduced lattice is constructed with the computation of a dictionary. Second, this dictionary is used to construct an unsupervised ordering by nonlinear dimensionality reduction. Third, this ordering is extended to all the points of the initial lattice by the Nyström extension, and the complete lattice is obtained. In Sections 4 and 5 we show how the proposed approach can be modified to either construct supervised orderings or adapt the ordering to several images. Section 6 considers the case of associating patches vectors to pixels and shows how our approach can be naturally used to obtain an innovative patch-based formulation of morphological operators. Last section concludes. The interest of the approach is illustrated all throughout

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the paper with various experiments and comparisons with state-of-the-art approaches.

2. Complete lattices in R^n

Mathematical morphology is a nonlinear approach to image processing that relies on a fundamental structure, the complete lattice $(\mathcal{L}, \leqslant)$ [13]. The complete lattice theory is widely accepted as the appropriate algebraic basis for MM. If this has the advantage of unifying previous approaches developed for binary and grayscale morphology, complete lattices also make it possible to generalize the fundamental concepts of morphological operators to a wider variety of image types.

2.1. Orderings

Since in complete lattices the concept of order plays a central role, we begin by recalling its key properties. Given $x, y, z \in \mathcal{A}$, a binary relation R on a set $\mathcal{A}$ is:

- reflexive if xRx;
- antisymmetric if xRy and $yRx \Rightarrow x = y$;
- transitive if xRy and $yRz \Rightarrow xRz$;
- total if xRy or yRx.

The binary relation R is a *pre-ordering* if R is reflexive and transitive. R is a *partial ordering* if R is an antisymmetric pre-ordering. Finally, R is a *total ordering* if it is a total partial ordering. Barnett [12] has proposed to classify ordering relations that operates on general vectors $\mathbf{v} = (v_1, \ldots, v_n)^T$ (i.e., the set $\mathcal{A}$ is $\mathbb{R}^n$) into four groups: marginal (M-ordering), conditional (C-ordering), partial (P-ordering) and reduced (R-ordering).

M-orderings. Orderings are performed on every component of the given vectors leading to a component-wise ordering:

$$\forall \mathbf{v}, \mathbf{v}' \in \mathbb{R}^n, \mathbf{v} \leq_M \mathbf{v}' \iff \forall i \in \{1, \ldots, n\}, v_i \leqslant v'_i. \tag{1}$$

Such an ordering is a *partial ordering*.

C-orderings. Vectors are ordered by means of their marginal components:

$$\begin{aligned} &\forall \mathbf{v}, \mathbf{v}' \in \mathbb{R}^n, \mathbf{v} \leq_C \mathbf{v}' \iff \exists i \in \{1, \ldots, n\}, \\ &\quad (\forall j < i, v_j = v'_j) \wedge (v_i \leqslant v'_i). \end{aligned} \tag{2}$$

The most well-known C-ordering is the lexicographic ordering that is a *total ordering* [14].

P-orderings. The ordering partitions the given vectors into equivalence classes with respect to rank or extremeness [12]. The most popular is the aggregated distance ordering that consists in associating each vector with the sum of its distances from the other vectors of a family $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$:

$$\forall \mathbf{v}, \mathbf{v}' \in \mathbb{R}^n, \mathbf{v} \leq_P \mathbf{v}' \iff \sum_{k=1}^{n} d(\mathbf{v}, \mathbf{v}_k) \leqslant \sum_{k=1}^{n} d(\mathbf{v}', \mathbf{v}_k). \tag{3}$$

Such an ordering is a *total pre-ordering*.

R-orderings. Vectors are first reduced to scalar values using a mapping $h : \mathbb{R}^n \to \mathbb{R}$. Vectors are then ordered with respect to the scalar order of their projection:

$$\forall \mathbf{v}, \mathbf{v}' \in \mathbb{R}^n, \mathbf{v} \leq_R \mathbf{v}' \iff h(\mathbf{v}) \leqslant h(\mathbf{v}') \tag{4}$$

Two main families of mappings h can be defined wether they are based on distances or projections [2]. According to the chosen transformation it is possible to obtain a *total pre-ordering* (h non-injective) or even a *total ordering* (h injective) [15].

Now that we have presented the concept of orderings, we can introduce the concept of complete lattices.

2.2. Complete lattices

A *partially order* set $\mathcal{A}$ is a set associated with a binary relation R that is reflexive, antisymmetric and transitive. To simplify the further notations, we will replace R by $\leqslant$.

In a partially ordered set $\mathcal{A}$, the least majorant $\vee\mathcal{X}$ (called supremum) of a subset $\mathcal{X} \subseteq \mathcal{A}$ is defined as an element $\mathbf{v}_0 \in \mathcal{A}$, such that: (1) $\mathbf{v}_i \leqslant \mathbf{v}_0$, $\forall \mathbf{v}_i \in \mathcal{X}$, and (2) if $\forall \mathbf{v}_i$, $\mathbf{v}_j \in \mathcal{X}$, such that $\mathbf{v}_i \leqslant \mathbf{v}_j \leqslant \mathbf{v}_0$, then $\mathbf{v}_j = \mathbf{v}_0$. One defines the greatest minorant $\wedge\mathcal{A}$ (called infimum) of $\mathcal{X}$ dually. Additional information can be found in [16,17].

A partially ordered set $\mathcal{A}$ is an *inf semi-lattice* (resp. *sup semi-lattice*) if every two-element subset $\{\mathcal{X}_1, \mathcal{X}_2\}$ in $\mathcal{A}$ has an infimum $\mathcal{X}_1 \wedge \mathcal{X}_2$ (resp. a supremum $\mathcal{X}_1 \vee \mathcal{X}_2$) in $\mathcal{A}$. If $\mathcal{A}$ is both an inf and a sup lattice, then it is called a lattice.

Finally, a lattice is called a *complete lattice* when every non-empty subset $\mathcal{X} \subseteq \mathcal{A}$ has an infimum $\wedge\mathcal{X}$ and a supremum $\vee\mathcal{X}$.

2.3. MM and complete lattices

It has been shown in [13] that any mathematical morphology operator must operate into the complete lattice structure of the object space. A space $\mathcal{L}$ endowed with a (partial or total) ordering relation $\leqslant$ is called a complete lattice [18], and is denoted by $(\mathcal{L}, \leqslant)$. As this was exposed in the previous section, this means that every non-empty subset $\mathcal{P} \subseteq \mathcal{L}$ has both an infimum $\wedge\mathcal{P}$ and a supremum $\vee\mathcal{P}$. Following the notation of [7], we say that the smallest element (minimum) $\mathbf{v}_k \in \mathcal{L}$ is an element contained in all others elements of $\mathcal{L}$, that is, $\mathbf{v}_l \in \mathcal{L} \Rightarrow \mathbf{v}_k \leqslant \mathbf{v}_l$. We denote the minimum of $\mathcal{L}$ by $\perp$. Equivalently, the largest element (maximum) $\mathbf{v}_k \in \mathcal{L}$ is an element that contains every element of $\mathcal{L}$, that is, $\mathbf{v}_l \in \mathcal{L} \Rightarrow \mathbf{v}_l \leqslant \mathbf{v}_k$. We denote the maximum of $\mathcal{L}$ by $\top$.

In this context, functions are modeled by mapping their domain space Ω, into a complete lattice $\mathcal{L}$, i.e., $f : \Omega \to \mathcal{L}$. Within this model, morphological operators are represented as mappings between complete lattices in combination with matching patterns called structuring elements that are subsets of Ω.

We call a *dilation* an operator $\delta : \mathcal{L} \to \mathcal{L}$ that commutes with the supremum and preserves $\top$ the lowest element of $\mathcal{L}$, i.e., δ is a dilation iff for every collection $\{\mathbf{v_i}\}_{i\in\mathcal{I}}$ of elements of $\mathcal{L}$:

$$\delta(\vee_{i\in\mathcal{I}}\mathbf{v_i}) = \vee_{i\in\mathcal{I}}\delta(\mathbf{v_i}), \tag{5}$$

and $\delta(\perp) = \perp$.

Similarly, we call *erosion* an operator $\epsilon : \mathcal{L} \to \mathcal{L}$ that commutes with the infimum and preserves $\perp$, the maximum of $\mathcal{L}$, i.e., ϵ is an erosion iff for every collection $\{\mathbf{v_i}\}_{i\in\mathcal{I}}$ of elements of $\mathcal{L}$:

$$\epsilon(\wedge_{i\in\mathcal{I}}\mathbf{v_i}) = \wedge_{i\in\mathcal{I}}\epsilon(\mathbf{v_i}), \tag{6}$$

and $\epsilon(\top) = \top$. As quoted in [19], dilation and erosion basically rely on three concepts: a ranking scheme, the extrema derived from this ranking and finally the possibility of admitting an infinity of operands (i.e., the two first are the ingredients of a complete lattice).

For any erosion ϵ, we can find a unique dilation δ such that $\forall \mathbf{v_i}, \mathbf{v_j} \in \mathcal{L} : \delta(\mathbf{v_j}) \leqslant \mathbf{v_i} \iff \mathbf{v_j} \leqslant \epsilon(\mathbf{v_i})$. A pair of erosion and dilation satisfying the above relation is called an adjunction. Given an adjunction (ϵ, δ) on a complete lattice, the following results can be easily proven [20]: (1) $\epsilon\delta \geqslant \mathbf{I}$ and $\delta\epsilon \leqslant \mathbf{I}$, (2) $\epsilon\delta\epsilon = \epsilon$ and $\delta\epsilon\delta = \delta$, (3) $\phi = \epsilon\delta$ is an opening, (4) $\gamma = \delta\epsilon$ is a closing.

To conclude, if one want to perform morphological operators on some data, one has first to look for a complete lattice for the set of values of the data since the ordering of the lattice enables to compare its elements. For example, if we consider the classical case of gray-level images $f : \Omega \to \mathbb{R}$, the corresponding complete lattice is $(\mathbb{R}, \leqslant)$ with $\leqslant$ the usual comparison operator in $\mathbb{R}$. However, if we now consider multivalued images $f : \Omega \to \mathbb{R}^n, n > 1$, it becomes problematic to find an ordering relation for the vectors of $\mathbb{R}^n$, due