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Moment invariants under similarity transformation

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ABSTRACT

This paper presents a method of deriving moment invariants under similarity transformation. The invariants sets are algebraically independent and complete. The method can be adopted for arbitrary dimension and order. This paper also presents some explicit formulas for computing invariant sets' dimension, which reveals how the number of invariants increases with moments' order and dimension elevating.

1. Introduction

For the advantage of shape characterization, moment invariants play an important role in a lot of applications among pattern recognition communities. They were first proposed by Hu in 1965 [1]. Since then a lot of effort has been devoted to this field, especially schemes for deriving new invariants. Li [2] and Wong [3] elevated the invariant order to nine and five, respectively. Mostafa and Psaltis [4] concentrated on the evaluation of invariants. In addition to the moments adopted by Hu, there are many other types of moments defined by Teh and Chin [5], which include Legendre moments, Zernike moments, pseudo-Zernike moments, and complex moments.

So lots of moment invariants are constructed based on these moments [6], Like invariants based on the Zernike moments [7,8], Legendre moments [9], Tchebichief moments [10] and Fourier-Mellin moments [11]. Lots of promising results are obtained by implementing these invariants in many fields of pattern recognition. Suk and Flusser also discussed infinite-term moment invariants for projective moment invariants [12].

For recognition of three-dimensional objects, three-dimensional moment invariants were derived based on algebraic invariants [13], tensor algebra [14], eigenvalues of the special matrices composed of moment [15] which will also be fully discussed in this paper, and group-theoretic technique [16]. Reiss proved that some invariants among them are just special cases of his descriptors [14]. In order to extend the dimension and order to be arbitrary, Mamistvalov presented a generalized fundamental theorem for moment invariants [17]. He proved that the moment invariants can be derived by using algebraic invariants. His work also for the first time makes it possible to analyze not only scenes consisting of polygons and polyhedra, but also scenes consisting of geometric figures and solids with curved contours and surfaces, respectively. Flusser et al. [18] gave the construction of two dimensional and three dimensional blur rotation moment invariants based on group theory, which first presented the blur invariants. Xu and Li [19] derived geometric moment invariants under similarity transformation. Their method is based on invariant geometric primitives such as distance, area, and volume. It has good extensibility in aspects of dimensions of spaces and orders of moments. But they also pointed out that their invariants might not be independent due to the complicated relations between geometric cores.

While much success has been demonstrated, some issues remain to be addressed. As the previous work mainly derived moment invariants based on the sufficient conditions, independence of invariant sets cannot be guaranteed. Flusser constructed two-dimensional moment invariant basis under rotation transformation. He proved it is algebraically independent and complete [20]. It is a pioneer work in the research of moment invariants' independence. However, it is difficult to extend this method to derive independent three dimensional invariants, although the idea of constructing rotation invariants is similar to that in their previous work [18]. We analyzed the necessary and sufficient conditions of moment invariants and derived independent and complete invariant sets of arbitrary dimension and order. But it is difficult to find the invariants because of the process for the

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solution of the parameter equations. Moreover, the invariant sets are merely linearly independent [21]. Hickman presented a sound method for constructing functionally independent affine invariants. However, his method cannot be used for deriving three dimensional affine invariants [22]. Suk and Flusser generated the two dimensional affine invariant set by graph method [23]. Then they eliminated linearly independent and polynomial dependent step by step. As they pointed out their algorithm had the drawback caused by higher-order dependencies. It means that the invariant sets may still be algebraically dependent after the eliminating operation. The dependency must be eliminated by multi steps.

The moment invariants discussed in this paper are mainly the invariants under similarity transformation, denoted as SMI. We also introduced some methods for more general moment invariants. Moreover, in this paper linearly independent SMI sets' being generated only needs the integral of the translating matrices for moments. Then they are simplified into algebraically independent sets by using Gröbner basis. The invariant sets are also complete. Besides, our method can be used for any dimension and order. Another advantage of our method is that all the algebraic relationships of the invariants can be found by only one step.

Another contribution of this paper is that we present two explicit formulas for computing the moment invariants sets' dimension. These invariants are invariant to rotation, denoted as RMI. We first constructed the projection matrix of RMIs' space based on Haar measure. Then we derived the formulas by using the eigenvalues of the projection matrix. One advantage of the formulas is that the dimension can be found before the invariants set are constructed, the other is it reveals how the number of RMI increases according to the increasing of the moments' dimension and order.

The rest of this paper is organized as follows: Section 2 gives a brief introduction of Haar measures and methods to compute them on special orthogonal groups; Section 3 presents the methods for dimension computing and construction of moment invariants; Section 4 carries out some experiments; Section 5 contains some concluding remarks and directions for future research.

2. Haar measure

This section first introduces the Haar measure on compact group. Then it describes the method to compute the Haar measures on special orthogonal groups.

Haar measure is an invariant integration defined on subsets of locally compact topological groups. This measure is invariant to the change of variables caused by group multiplication. A Haar measure on B is a measure $\chi:\Sigma\rightarrow[0,+\infty)$, with Σ a σ-algebra containing all Borel subsets of B, such that $\chi(B)=1$ and $\chi(\delta S)=\chi(S)$ $(\delta S=\{\delta\alpha|\alpha\in S\})$ for all $\delta\in B$ and $S\in\Gamma$. We may associate to the measure χ a bounded linear function denoted as $\int_B f(\delta)d\delta$. $\int_B f(\delta)d\delta$ is the Haar integral which is the Lebesgue integral based the Haar measure. An unique $\int_B f(\delta)d\delta$ must exist, such that $\forall\delta_i\in B$, $\int_B f(\delta)d\delta=\int_B f(\delta_i\delta)d\delta$ and $\int_B 1d\delta=1$, if $f:B\rightarrow\mathfrak{R}$ is a continuous non-constant function and B is compact group [24].

In order to compute $\int_B f(\delta)d\delta$ we need to find a function $\varpi:\mathfrak{R}^n\rightarrow\mathfrak{R}$ and a homeomorphism $\phi:\mathfrak{R}^n\rightarrow B$ which obeys $\phi(0)=I$, such that $\int_B f(\delta)d\delta=\int_W f(\phi(x_1,\dots,x_n))\varpi(x_1,\dots,x_n)dx_1,\dots,dx_n$ for all continuous functions $f:B\rightarrow\mathfrak{R}$, where W is an open neighborhood in $\mathfrak{R}^n$. The measure $\varpi(x_1,\dots,x_n)dx_1\dots dx_n$ represents the Haar measure in local coordinates [25].

Lots of important groups are compact, such as the orthogonal group, the unary group, and the symplectic group. Among them, one of the most important groups for moment invariants is the n dimensional special orthogonal group, denoted as $SO(n)$, for it represents the n dimensional rotation. $SO(n)$ is a smooth manifold. Its elements are the $n\times n$ matrices R so that $RR^T=id$ where id is the identity matrix with the same dimension of R. Since RR^T is symmetric, $RR^T=id$ only contain $n(n-1)/2$ equations. So the dimension of $SO(n)$ is equivalent to $n(n-1)/2$.

The following is the Haar measures in local coordinates for $SO(2)$ and $SO(3)$.

For $SO(2)$, $\phi:\theta\rightarrow\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$, $0\le\theta<2\pi$, then

$$\int_B f(x)dx=\frac{1}{2\pi}\int_0^{2\pi}f(\phi(\theta))d\theta.$$

For $SO(3)$, the elements of matrices can be defined by the functions of Euler angles$(\varphi_1,\theta,\varphi_2)$, $0\le\varphi_1,\varphi_2<2\pi$, $0\le\theta<\pi$, which is

$$\phi:(\varphi_1,\theta,\varphi_2)$$
$$\rightarrow\begin{pmatrix}\cos\varphi_2 & -\sin\varphi_2 & 0\\ \sin\varphi_2 & \cos\varphi_2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\varphi_1 & -\sin\varphi_1 & 0\\ \sin\varphi_1 & \cos\varphi_1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Then

$$\int_B f(x)dx=\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}f(\phi(\varphi_1,\theta,\varphi_2))\sin\theta\,d\varphi_1 d\theta\,d\varphi_2.$$

3. SMI sets' construction

3.1. Relationship between moments

In this section, we give some definitions for describing the relation between moments. In order to ascertain subjects for further elaboration, the definition of geometric moments is given first.

Considering that $m_{p_1\dots p_n}$ represents the n dimensional $(p_1+\dots+p_n)$ th order geometric moment of a piecewise continuous density function $h(x_1,\dots,x_n)$. It can be defined in terms of Riemann integrals as:

$$m_{p_1\dots p_n}=\int\dots\int_\infty x_1^{p_1}\dots x_n^{p_n}h(x_1,\dots,x_n)dx_1\dots dx_n.$$

Given a group with a linear action on $h(x_1,\dots,x_n)$, the induced action on moments is as same as the linear action on polynomials. It is introduced in details in the following part.

Consider the vector $x\in\mathfrak{R}^n$, define

$$x^{(p)}=\begin{pmatrix}{x_1}^p\\ {x_1}^{p-1}x_2\\ \dots\\ {x_n}^p\end{pmatrix}$$

where the ordering of $x_1^{p_1}\dots x_n^{p_n}$ is lexigraphic of p_i's, $p_i\in\{0,1,\dots p\}$ and $\sum_{i=1}^{n}p_i=p$. Then there exists a matrix denoted $\alpha^{(p)}$ such that $(\alpha x)^{(p)}=\alpha^{(p)}x^{(p)}$. The same relationship exists in the moments, as shown in the following proposition [17,22].

Proposition. *Let* $m'_{p_1\dots p_n}=\int\dots\int_\infty x_1'^{p_1}\dots x_n'^{p_n}h(x'_1,\dots,x'_n)dx_1\dots dx_n$, *where* $(x'_1,\dots,x'_n)^T=\delta(x_1,\dots,x_n)^T$, δ *is a matrix and* $|\delta|=1$. *Then* $(m'_{p_1\dots p_n})^T=\delta^{(p)}(m_{p_1\dots p_n})^T$ *where* $\sum_{i=1}^{n}p_i=p$ *and the ordering of* $m_{p_1\dots p_n}$ *is lexigraphic of* p_i*'s.*

For example: if $\delta=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$, $0\le\alpha<2\pi$, We have

$$(m'_{20},m'_{11},m'_{02})=\delta^{(2)}(m_{20},m_{11},m_{02})^T$$
$$=\begin{pmatrix}\cos^2\alpha & -2\cos\alpha\sin\alpha & \sin^2\alpha\\ \cos\alpha\sin\alpha & \cos^2\alpha-\sin^2\alpha & -\cos\alpha\sin\alpha\\ \sin^2\alpha & 2\cos\alpha\sin\alpha & \cos^2\alpha\end{pmatrix}(m_{20},m_{11},m_{02})^T.$$

If there exists a group B whose elements consists and only consists of the matrices δs, it is easy to prove that there must exist a group whose elements consist and only consist of the matrices $\delta^{(p)}$s, denoted as $B^{(p)}$.