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Reducing aliasing in images: a PDE-based diffusion revisited

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ABSTRACT

In this paper, we introduce a new diffusion algorithm that can be used for reducing aliasing on both step edges and lines. It derives from the diffusion model of Perona and Malik, and works as an adaptive level-curve method in which diffusion is carried out in the normal direction of the gradient for step edges, while the eigenvalues of the Hessian matrix are used for lines. To get sharp images, we use high-pass filters to preserve as much as possible the high frequency content while diffusing. Experimental tests using grayscale and colour images show that our algorithm efficiently reduces aliasing.

1. Introduction

In signal processing, aliasing refers to an effect that causes different signals to become indistinguishable (or *aliases* of one another) when sampled [1]. More commonly in image processing, aliasing is found in the form of poorly pixelized images in which stairlike edges appear where they should normally be smooth. These stairlike edges are generally called *jaggies* or *aliasing*. An example of an image featuring aliasing can be seen in Fig. 1 where the black ellipses show some aliasing. Aliasing is frequently encountered in resampling algorithms, and severe aliasing is particularly found when using the nearest neighbour resampling algorithm [2].

Aliasing phenomena can be explained using the sampling theory. Indeed, Nyquist–Shannon's theorem asserts that the uniformly spaced discrete samples of a signal that is bandlimited are a complete representation of the signal if the bandwidth is less than half the sampling rate [3]. When Nyquist–Shannon's conditions are not satisfied, aliasing occurs. Thus, to reduce aliasing, a linear low-pass filter is sometimes used in order to remove frequencies above half the sampling rate. However, the output image is blurred. To overcome that issue, authors have proposed to selectively blur pixels that are concerned by aliasing [4]. Diffusion process is among the best strategies found in the literature for selectively blurring an image. Moreover, it is used for image denoising, image deblurring and image enhancement [4–7]. Diffusion is analytically defined using the following partial differential equation (PDE) known as the diffusion equation:

$$\begin{cases} \frac{\partial f(x,y,t)}{\partial t} = div\left[\lambda(x,y,t)\nabla f(x,y,t)\right] \\ f(x,y,0) = f_0(x,y) \end{cases} \tag{1}$$

where $f(x,y,t)$ is the density of the diffusing material at pixel location (x,y) and time t, $\lambda(x,y,t)$ the diffusion coefficient at location (x,y) and time t, ∇ the gradient operator and *div* the divergence operator. At the initial time $t=0$, f is equal to f_0, which is the starting point of the diffusion process. Perona and Malik [8] have used diffusion for removing noise in images, and PDE (1) can also be used for reducing aliasing in images. In the next lines, we consider aliasing in the case of step edges. In Section 5, we will study the case of lines. For a better diffusion that does not oversmooth images, the diffusivity λ is not constant and depends on x, y and t, which gives an anisotropic diffusion. If λ does not depend on f, the diffusion is linear; in the other case, it is nonlinear. Some expressions given to λ are as follows:

$$\lambda(x,y,t) = \frac{1}{1+\left(\frac{\|\nabla f(x,y,t)\|}{\beta}\right)^2}$$

or

$$\lambda(x,y,t) = \exp\left(-\frac{\|\nabla f(x,y,t)\|^2}{2\beta^2}\right) \tag{2}$$

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In (2), the coefficient β is used to control the diffusion factor. We can observe that when the gradient is large (which corresponds to edge points), λ is small; in the opposite, λ is high when the gradient is small (which corresponds to smooth or homogenous regions). Note that the diffusivity $\lambda(x,y,t)$ is chosen to be a decreasing function of the magnitude of the gradient in order to sharpen edges during the diffusion process [8]. In [9,10], more information can be found on the selection of the diffusivity coefficients. In the rest of the paper, we use $\lambda(x,y,t)$ as the first expression in (2). The PDE given in (1) tends to create images, which are not pleasant visually [11]. In fact, even if it is efficient for blurring and reducing noise, it is not efficient for reducing aliasing found on edges (see for example the borders of the hat in the front in Fig. 2), and we propose a new solution, which exploits the local curvature of pixels around edges to efficiently reduce edge aliasing while preserving the uniform regions.

The model that we propose makes use of an inverse diffusivity $1-\lambda(x,y,t)$, and it can be seen as both a simplification and an enhancement of the diffusion equation of Perona and Malik. In fact, we examine the diffusion PDE of Perona and Malik, and we deduce a PDE adapted for reducing aliasing. The rest of the paper is organized as follows: in Section 2, we describe the weakness of Perona and Malik's PDE in reducing aliasing. In Section 3, we introduce our new model for reducing aliasing in step edges in the case of grey-level images, and in Section 4 we present some experimental results. In Section 5, we extend our model for reducing aliasing in images containing lines. In Section 6, we describe the application of our model in the case of colour images.

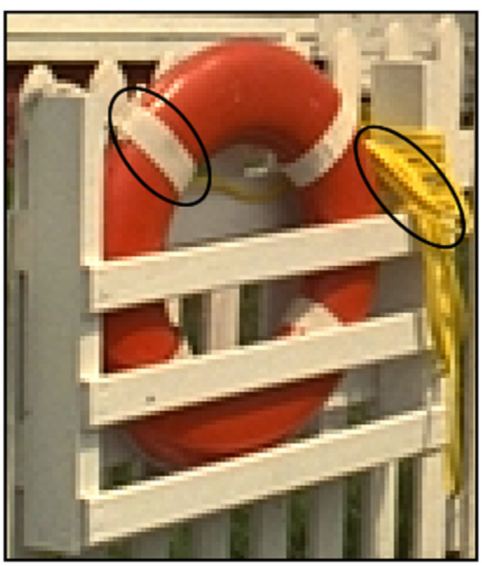

Fig. 1. Image containing aliasing (jaggies).

2. Inefficiency of Perona and Malik's PDE in reducing aliasing

By decomposing the divergence operator, (1) is equivalent to

$$\begin{cases} \frac{\partial f(x,y,t)}{\partial t} = \lambda(x,y,t)\Delta f(x,y,t) + \nabla\lambda(x,y,t)\cdot\nabla f(x,y,t) \\ f(x,y,0) = f_0(x,y) \end{cases} \tag{3}$$

where $\cdot$ denotes the scalar product, and Δ is the Laplacian operator. We will now show that this PDE is not suitable for reducing aliasing. In [12], it is shown that

$$\Delta f = \frac{\partial^2 f}{\partial n^2} + \frac{\partial^2 f}{\partial n_\perp^2} = \frac{\partial^2 f}{\partial n^2} + \kappa\|\nabla f\| \tag{4}$$

where n is the direction of the gradient, $n_\perp$ the normal direction of the gradient, κ the curvature along the underlying edge, which is given by (in practice, 1 is added to the denominator to avoid null values, which can occur in constant regions for example)

$$\kappa = \frac{f_x^2 f_{yy} - 2f_x f_y f_{xy} + f_y^2 f_{xx}}{(f_x^2+f_y^2)^{3/2}} \tag{5}$$

Thus, the Laplacian is decomposed into the change of the magnitude of the gradient vector along the direction of the gradient and the change of direction of the gradient vector, multiplied by the gradient magnitude, along the direction perpendicular to the gradient. The term $\partial^2 f/\partial n_\perp^2$ is related to the curvature of the underlying edge [13], and is equal to zero only if the underlying edge is straight and theoretically infinite at the junction points. Note that the second directional derivative along the gradient is [14]

$$\frac{\partial^2 f}{\partial n^2} = (1+f_x^2+f_y^2)^{3/2} K_n \tag{6}$$

where K_n is the normal curvature along the gradient [15], which is computed as follows:

$$K_n = \frac{f_x^2 f_{xx} + 2f_x f_y f_{xy} + f_y^2 f_{yy}}{(f_x^2+f_y^2)(1+f_x^2+f_y^2)^{3/2}} \tag{7}$$

As $\nabla\lambda(x,y,t)\cdot\nabla f(x,y,t)$ represents a scalar product, and by substituting (4) in (3), we derive:

$$\begin{cases} \frac{\partial f(x,y,t)}{\partial t} = \lambda(x,y,t)\frac{\partial^2 f(x,y,t)}{\partial n^2} + \left[\lambda(x,y,t)\kappa + \|\nabla\lambda(x,y,t)\|\cos\phi(x,y,t)\right]\|\nabla f(x,y,t)\| \\ f(x,y,0) = f_0(x,y) \end{cases} \tag{8}$$

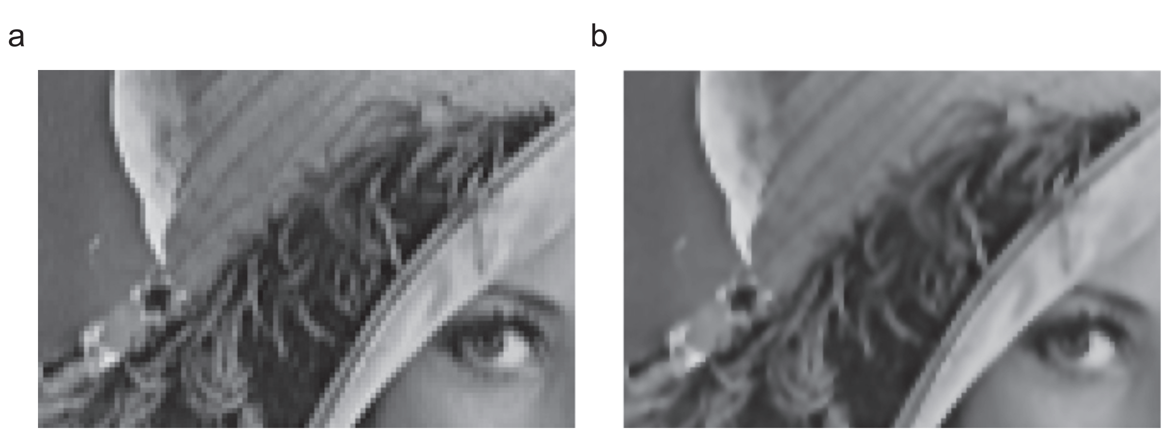

Fig. 2. (a) Part of an aliased image and (b) resulting image obtained after 8 iterations using the algorithm of Perona and Malik.