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Connectivity calculus of fractal polyhedrons

Helena Molina-Abril [a,*], Pedro Real [a], Akira Nakamura [b], Reinhard Klette [c]

[a] Universidad de Sevilla, Spain
[b] Hiroshima University, Japan
[c] The University of Auckland, New Zealand

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ABSTRACT

The paper analyzes the connectivity information (more precisely, numbers of tunnels and their homological (co)cycle classification) of fractal polyhedra. Homology chain contractions and its combinatorial counterparts, called homological spanning forest (HSF), are presented here as an useful topological tool, which codifies such information and provides an hierarchical directed graph-based representation of the initial polyhedra. The Menger sponge and the Sierpiński pyramid are presented as examples of these computational algebraic topological techniques and results focussing on the number of tunnels for any level of recursion are given. Experiments, performed on synthetic and real image data, demonstrate the applicability of the obtained results. The techniques introduced here are tailored to self-similar discrete sets and exploit homology notions from a representational point of view. Nevertheless, the underlying concepts apply to general cell complexes and digital images and are suitable for progressing in the computation of advanced algebraic topological information of 3-dimensional objects.

1. Introduction

The Polish mathematician Waclaw Sierpiński described properties of a self-similar planar set (see [26]), today known as the *Sierpiński sieve*. A generalization into 3-dimensional space leads to the *Sierpiński pyramid*. The Austrian mathematician Karl Menger also studied a recursively defined set, today known as *Menger sponge* (see [16] and Fig. 1), when discussing the notion of topological dimension.

In this paper we consider recursively defined polyhedra, called *fractal sets*. Recursion details are given below. The Menger sponge or Sierpiński pyramid are examples of such fractal sets.

Working within a semi-continuous context, we use the notion of a *minimal spanning tree* (MST) of a finite set of points in $\mathbb{R}^n$, as introduced in [22,23], to provide topological information of the fractal underlying set, up to a numerically computable resolution.

In particular, connectedness, disconnectedness and the number of connected components with non-zero diameter are properties that are identified and calculated in [23]. A logical question that arises here when investigating ways to formalize the relationship between the homology of a set of $\mathbb{R}^3$, and the homology of a finite point-set approximation of it, is how to distinguish between simply-connected sets and those with holes (i.e. tunnels or cavities).

Within the semi-continuous context of cell complexes, there are two approaches for the computation of comprehensive homological information:

1. *The (co)differential approach:* Here, only one 2-nilpotent linear map (the canonical differential operator of the cell complex) is involved, and linear algebra for reducing matrices into a Smith Normal Form is exhaustively used (see e.g [12,19]).
2. *The integral approach:* Here, two 2-nilpotent linear maps (differential and integral operators) are involved. In the integral approach, one constructs a degree $+1$ linear map (integral operator) which records the information of the algebraic homological deformation process, reducing the whole cell complex to a minimal homological expression (see e.g. [10]).

In particular, using the differential and integral operators, one can determine homology groups among other topological properties.

For example, the method of computing connected components using the spanning forest of the 1-skeleton of the cell complex is a relevant example of integral homological computation. We apply here an integral homological approach, allowing a quantitative analysis of high degrees of connectivity of cell complex versions of fractals, for an arbitrary level of recursion. We mainly focus on the

* Corresponding author.
E-mail addresses: habril@us.es (H. Molina-Abril), real@us.es (P. Real), an1206@ad.cyberhome.ne.jp (A. Nakamura), r.klette@auckland.ac.nz (R. Klette).

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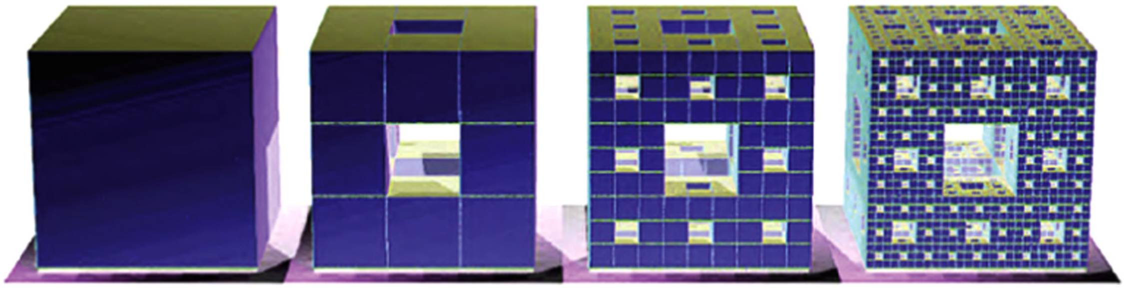

Fig. 1. Menger sponge at recursion levels 0–3, from left to right (published by *Solkoll* in 2005 in the public domain; here shown color-inverted). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

number of tunnels and cavities, but the techniques developed here can be employed to create a *calculus* with cycles and co-cycles within the fractals.

A cycle calculus allows us, for example,

1. to topologically classify any cycle or co-cycle within the object,
2. to determine whether a cycle is contractible within the object and, in the positive case, to obtain a *geometric deformation* that reduces the cycle into a point,
3. to topologically transform, if possible, one cycle into another within the fractal,
4. to determine a shortest path between two points in the given fractal, considering a defined distance function, and
5. to compute (co-)cyclic operations having (co-)cycles of the objects as input or output.

Starting with a hierarchical directed graph representation, it is possible to proceed with cycle-calculus on fractals. This calculus has been introduced in [17,18], by defining coordinate-based *forests* (i.e. graphs which are in general a set of trees). These direct graphs extent to higher dimensions spanning forests as known from labelling of connected components (0th-homology group) of a digital object. Due to this fact, they are called *homological spanning forests* (HSFs). These structures can also be generalized for applications in the domain of tree–cotree decompositions of combinatorial surfaces; see [6] and [7].

In this paper we recall the definition of HSFs, give relevant examples for fractal sets, and manipulate their associated chain-homotopy equivalence for computing topological properties. For developing this approach, different methodologies and theories have been combined, such as discrete Morse theory (see [8]), effective homology (see [25]), AT-model theory (see [10]), and homological algebra (see [17]).

The techniques introduced here are tailored to fractals, but the underlying concepts also apply to digital images and data. The development of new topological representations is essential for advancing in solid and physical modelling [3,4]. It appears to be a feasible short-term goal to design a novel mapping from mathematical solid models to actual computer representations for extensive geometric data as common in 3D imaging, using the HSF techniques as developed. See the reported experiments at the end of the paper.

Furthermore, digital topology has various applications in remote sensing, computer vision, lossless and fractal compression, and algorithmic pattern recognition. Because our method applies to fractal and non-fractal cell complexes, the work presented here is susceptible to be adapted for its use in the previously mentioned applications.

The paper is structured as follows: Section 2 introduces HSFs and related basic definitions. Section 3 applies the introduced topological framework to fractal polyhedra in general. The example of the Menger sponge is discussed in Section 4, and those of the Sierpiński pyramid in Section 5. Experiments are shown in Section 6. Section 7 concludes.

2. Combining homological with homotopical information

We determine the *representative cycles* of tunnels in fractal structures which are embedded into $\mathbb{R}^3$. By using HSF structures, we classify any closed curve in the fractal structure in terms of homology generators working with coefficients in the field $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$.

A *cell complex O* consists of *i*-cells of dimension *i*, for some $i \in \mathbb{N}$. Given a finite convex cell complex embedded in $\mathbb{R}^3$, it is possible to construct the canonical chain complex associated to a cell complex *O* and its homology, called a *strong deformation retract* in [11], a *reduction* in [25], or a *chain contraction* in [5]. In fact, a chain contraction of this kind, called homology chain-integral contraction (See [20]), condenses the $\mathbb{Z}/2\mathbb{Z}$ homological information of the cell complex *O* and it can be completely described by only using two chain operators. One of them is the (unique) chain boundary operator for *O*. The other is a (non-unique) two-nilpotent chain homotopy called chain integral operator.

In order to take advantage of the geometric nature of the cell complex, we combine homological and homotopical information of *O*, for constructing the previous homology chain contraction by using the combinatorial boundary operator of *O* and directed graphs whose nodes are the centroids of the (convex) cells. For example, if the homotopical information is given in terms of a gradient vector field *V* on the cell complex *O* (see [8,14]), it is straightforward to construct a chain contraction from the chain complex *O* to a smaller (in terms of numbers of basis generators) algebraic object. In the case having an optimal vector field *V*, this last object is the $\mathbb{Z}/2\mathbb{Z}$-homology graded module of *O*. Summing up, we search here for an "economical" combinatorial coding of a homology chain integral operator (also called, AT-model, [9]). This "coding" will be given here in terms of hierarchical coordinate-based directed graphs, called *homological spanning forest* (HSF). This paper discusses the constructions of HSF structures for some fractal sets, to be used for determining homology chain contractions.

The directed edges of an HSF are part of the frontier of the cell complex which consists of several "connected" *cell pairings*. A cell pairing is a directed edge which goes from an *i*-cell to an incident $(i+1)$-cell of the complex. Let us note that in a general HSF, two cell pairings could share the *i*-cell or the $(i+1)$-cell components. The links between these cell pairings are also directed edges from the $(i+1)$-cell tail of one cell pairing to the *i*-cell source of the other. When the HSF is derived from a optimal discrete gradient vector field *V* over *O* (that is, with a minimum number of critical cells), the cell pairings are disjoint pairs of incident cells. In this case, the connected directed graphs (DG for short) of the HSF can be described by the union of different non-trivial closed *V*-paths [8].