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Image reconstruction from continuous Gaussian–Hermite moments implemented by discrete algorithm

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A B S T R A C T

The problem of image reconstruction from its statistical moments is particularly interesting to researchers in the domain of image processing and pattern recognition. Compared to geometric moments, the orthogonal moments offer the ability to recover much more easily the image due to their orthogonality, which allows reducing greatly the complexity of computation in the phase of reconstruction. Since the 1980s, various orthogonal moments, such as Legendre moments, Zernike moments and discrete Tchebichef moments have been introduced early or late to image reconstruction. In this paper, another set of orthonormal moments, the Gaussian–Hermite moments, based on Hermite polynomials modulated by a Gaussian envelope, is proposed to be used for image reconstruction. Especially, the paper's focus is on the determination of the optimal scale parameter and the improvement of the reconstruction result by a post-processing which make Gaussian–Hermite moments be useful and comparable with other moments for image reconstruction. The algorithms for computing the values of the basis functions, moment computation and image reconstruction are also given in the paper, as well as a brief discussion on the computational complexity. The experimental results and error analysis by comparison with other moments show a good performance of this new approach.

1. Introduction

Today, in the image analysis, image storage and image transmission fields, there is a large requirement for image characterization, reconstruction and compression. The theory of statistical moments provides an interesting and useful alternative to series expansions for representing a real bounded function, so moments are widely used as one of the most common tools in a number of applications, especially in different fields of digital image processing and pattern recognition.

Historically, Hu has first introduced moment invariants from methods of algebraic invariants [1]. This is a set of seven nonlinear combinations of geometric moments proposed as image invariant global features, which are translation, rotation and scale independent. Hu's moment invariants are widely used for pattern description and recognition. Theoretically, the original function can be recovered without error using an infinite set of all its moments. So image reconstruction using its moments has become a challenging problem. Unfortunately, the geometric moments are not orthogonal, from which image reconstruction is practically quite difficult, although it is realizable [2,3]. Therefore, it is preferred to use the orthogonal moments defined in terms of a set of orthogonal basis functions. The first considerable work on this subject was published in 1980 by Teague, who had initially introduced Legendre moments and Zernike moments based, respectively, on continuous Legendre and Zernike polynomials [4]. Some other orthogonal moments were then proposed for image analysis.

Image reconstruction from the orthogonal moments is much easier than that from non-orthogonal moments. In fact, an image could be reconstructed as a simple summation of orthogonal polynomials, weighted by the moment values. However, there are two main sources of error involved in practical implementation: the discrete approximation of the continuous integral [5] and the transformation of the image coordinate system into the domain of polynomials [6]. The orthogonality of continuous basis functions will be destroyed because of numerical approximations. That is why the reconstruction is never perfect from the continuous orthogonal moments. In order to resolve this problem, the discrete orthogonal moments begin to be used since the beginning of the century. Mukundan proposed to use a set of discrete orthogonal moments based on the Tchebichef polynomials in a discrete variable for image reconstruction and he also offered the efficient method for their computation [7–9]. On the same principle, discrete Krawtchouk polynomials and other discrete orthogonal polynomials are successively suggested to be the basis functions for defining the

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discrete orthogonal moments, of which the implementation does not involve any numerical approximation [10–12]. Compared with the continuous orthogonal moments, the discretization error does not play a role in the reconstruction from the discrete orthogonal moments.

The principal goal of image reconstruction studies, especially the image reconstruction from only a limited number of its moments, is to resolve some practical problems such as that mentioned at the beginning of this section. Although the image can be perfectly reconstructed from all its discrete moments of a certain family as discrete Tchebichef one, yet it is not sure that the result from a limited number set is always the best. However, a higher quality of image reconstruction only from a limited number of moments makes this technique have a significant applicability.

In this paper we still focus our attention on image reconstruction from the continuous orthogonal moments. Another orthogonal moment named Gaussian–Hermite moment is proposed to be employed in image analysis [13], which has firstly been introduced to the astrophysics field in 1993 [14]. Gaussian–Hermite moment basis functions of different orders have different number of zero-crossings of which the distribution is more equidistant than other orthogonal moment functions. More specifically, its zero-crossings are distributed a little more densely in the vicinity of the origin than in the peripheral regions. This property is very important for image reconstruction resolution. Since their basis functions are much more smoothed, Gaussian–Hermite moments are thus less sensitive to noise. Moreover, the discretization produces less influence on the orthogonality than that of other continuous moments. So the reconstruction result from Gaussian–Hermite moments is much better than that from the other continuous orthogonal moments as Legendre moments and even than that from discrete Tchebichef moments. Our first work for image reconstruction from its Gaussian–Hermite moments is reported in Refs. [15,16]. The innovation of this paper includes two important points: how to determine the best scale parameter and how to improve the reconstruction results, both automatically. It is the solution of these two questions that makes image reconstruction from its Gaussian–Hermite moments become practically applicable and comparable with other orthogonal moments, even the discrete ones.

The paper is organized as follows: The general introduction of Gaussian–Hermite polynomials and their corresponding moments are defined in Section 2. Some aspects of computational complexity are given in the same section. Section 3 presents the discussion on the influence which the discrete implementation brings to orthonormality of the basis functions and how to estimate the best scale parameter. In Section 4, a normalization is suggested to be performed as post-processing in order to improve reconstruction results. Section 5 shows the experimental results with binary as well as gray-level images. The comparison in image reconstruction with different orthogonal moments is also given in this section. The paper ends with a brief conclusion and some remarks, which are presented in Section 6.

2. Gaussian–Hermite moments and image reconstruction

2.1. Hermite polynomials

Besides Legendre, Zernike, Laguerre and Tchebichef polynomials, another family of orthogonal polynomials is that of the Hermite polynomials $H_n(x)$, which are defined over the domain $(-\infty,\infty)$:

$$H_n(x) = (-1)^n \exp(x^2)\frac{d^n}{dx^n}\exp(-x^2). \tag{1}$$

These polynomials are orthogonal with respect to the weight function $\exp(-x^2)$. Their orthogonality is presented by

$$\int_{-\infty}^{\infty} \exp(-x^2) H_m(x) H_n(x)\,dx = 2^n n! \sqrt{\pi}\,\delta_{mn}, \tag{2}$$

where δ_{mn} is the Kronecker delta and

$$\delta_{mn} = \begin{cases} 0, & m \neq n, \\ 1, & m = n \end{cases} \tag{3}$$

Like other orthogonal polynomials that can be derived by the recurrence formula, Hermite polynomial can also be calculated by the following equation:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \text{ for } n \geq 1, \tag{4}$$

With $H_0(x)=1$ and $H_1(x)=2x$. It should be noted that Hermite polynomials are orthogonal but not orthonormal. In order to make them orthonormal we introduce the orthonormal Hermite polynomials which are the normalized versions of Hermite polynomials. The orthonormal Hermite polynomials, or Gaussian–Hermite polynomials, can be written in the following form:

$$\hat{H}_n(x) = (2^n n! \sqrt{\pi})^{-1/2} \exp(-x^2/2) H_n(x), \tag{5}$$

which keeps the orthonormality

$$\int_{-\infty}^{\infty} \hat{H}_m(x)\hat{H}_n(x)\,dx = \delta_{mn}. \tag{6}$$

2.2. Gaussian–Hermite moments

In Eq. (5), the factor $\exp(-x^2/2)$ represents a Gaussian envelope. Replacing x by (x/σ) to adjust the scale of Gaussian–Hermite polynomials and meanwhile making it satisfy Eq. (6), we can obtain the generalized Gaussian–Hermite polynomials with the scale parameter σ which in fact is the standard deviation for the Gaussian envelope:

$$\hat{H}_n(x/\sigma) = (2^n n! \sqrt{\pi}\sigma)^{-1/2} \exp(-x^2/2\sigma^2) H_n(x/\sigma). \tag{7}$$

Obviously Eq. (7) keeps the orthogonality because it satisfies the following equation:

$$\int_{-\infty}^{\infty} \hat{H}_m(x/\sigma)\hat{H}_n(x/\sigma)\,dx = \delta_{mn}. \tag{8}$$

By using the above Gaussian–Hermite polynomials as basis functions, Gaussian–Hermite moments of order (m,n) for a continuous two-dimensional function can be defined over the domain $(-\infty \leq x,\, y \leq \infty)$ as follows:

$$\eta_{mn} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{H}_m(x/\sigma)\hat{H}_n(y/\sigma) f(x,y)\,dx\,dy, \tag{9}$$

where $f(x,y)$ is the continuous two-dimensional image function.

Because the nth degree Hermite polynomial $H_n(x)$ has n different real roots over the interval $(-\infty,\ \infty)$, it is easy to demonstrate that the basis function of Gaussian–Hermite moment $\hat{H}_n(x/\sigma)$ has also n different real roots. Therefore, the nth order Gaussian–Hermite basis function will change its sign n times. In the frequency domain, Gaussian–Hermite basis functions have the same behavior of that in the spatial domain and comprise more and more oscillations when the order increases [17]. They contain thus more and more high frequencies. Moreover, from the viewpoint of spectral analysis, they can be looked as "quasi-band limited" functions. The Fourier transform of $\hat{H}_n(x/\sigma)$ is given as follows:

$$\begin{aligned}\mathcal{F}[\hat{H}_n(x/\sigma)] = {} & (-i)^n[\sigma/(2^n n!\sqrt{\pi})]^{1/2} \\ & \times \exp[-(2\pi\sigma f)^2/2] H_n(2\pi\sigma f),\end{aligned} \tag{10}$$

where $\mathcal{F}[]$ is the Fourier transform operator and $i=\sqrt{-1}$. Fig. 1