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Computing upper and lower bounds of rotation angles from digital images

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ABSTRACT

Rotations in the discrete plane are important for many applications such as image matching or construction of mosaic images. We suppose that a digital image A is transformed to another digital image B by a rotation. In the discrete plane, there are many angles giving the rotation from A to B, which we call admissible rotation angles from A to B. For such a set of admissible rotation angles, there exist two angles that achieve the lower and the upper bounds. To find those lower and upper bounds, we use hinge angles as used in Nouvel and Rémila [Incremental and transitive discrete rotations, in: R. Reulke, U. Eckardt, B. Flash, U. Knauer, K. Polthier (Eds.), Combinatorial Image Analysis, Lecture Notes in Computer Science, vol. 4040, Springer, Berlin, 2006, pp. 199–213]. A sequence of hinge angles is a set of particular angles determined by a digital image in the sense that any angle between two consecutive hinge angles gives the identical rotation of the digital image. We propose a method for obtaining the lower and the upper bounds of admissible rotation angles using hinge angles from a given Euclidean angle or from a pair of corresponding digital images.

1. Introduction

Rotations in the discrete plane are required in many applications for image computation such as image matching, construction of mosaic images [2]. For the moment, the most popular method to estimate the rotation angle is to approximate the rotation matrix by minimizing errors [2].

In the continuous plane, the Euclidean rotation is well defined and possesses the property of bijectivity. This implies that for two angles γ_1, γ_2 and a set of points A, if the Euclidean rotation with angle γ_1 applied to A gives the same result as the Euclidean rotation with angle γ_2 applied to A, then we have $\gamma_1 = \gamma_2$.

In the discrete plane, however, two different points can arrive at the same grid point after a discretization of the Euclidean rotation (DER). Because of this reason, two different angles can give the same rotation result for a set A of grid points.[1] In other words, we can define a set of *admissible rotation angles* (ARA) S such that any angle in S gives the same rotation result for A. Note that S depends on A. The two most interesting angles in S are the lower and the upper bounds because with only these two angles we can deduce the other angles in S. This paper aims to find these two angles from a given rotation angle or from two given corresponding sets of grid points. In order to identify the exact bounds, we should not involve any computation error. Thus, we work with discrete geometry tools which guarantee to avoid computation with real numbers. Moreover, because we assume that our data are discretized from continuous images of an object, we enforce the property that the discrete rotation[2] between two different sets of grid points gives the same result as DER.

Some work on discrete rotations already exists. The most widely used discrete rotation in the beginning is the CORDIC algorithm [3]. The CORDIC algorithm uses a sequence of fractions for an approximation of π. It multiplies/adds fractions in this sequence to approximate values of sine and cosine. Thus multiple approximations lead to little differences between results of DER and those of CORDIC. Andres [4,5] described some discrete rotations such as the rotation by discrete circles, the rotation by Pythagorean lines or the quasi-shear rotation. Computation executed during these rotations are exact. However, because they preserves the bijectivity, they cannot give the same results as DER.

On the other hand, Nouvel and Rémila [1] proposed another discrete rotation based on hinge angles, which gives the same results as

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[1]Accordingly, DER is not bijective.

[2] A discrete rotation is a rotation designed for the discrete space. It transforms a set of grid points into another set of grid points.

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DER. It is known that a sequence of hinge angles is a set of particular angles determined by a digital image in the sense that any angle between two consecutive hinge angles gives the identical rotated digital image. This means that hinge angles correspond to the discontinuity of DER. Nouvel and Rémila showed that each hinge angle is represented by an integer triplet, so that any discrete rotation of a digital image is realized only with integer calculation. Because their algorithm gives the same rotation results as DER, we see that hinge angles represented by integer triplets give sufficient information for executing any digital image rotation.

In this paper, we propose a discrete method for finding the lower and the upper bounds of ARA. Our method uses hinge angles because we can obtain the same result as DER and because they allow exact computations. The input data of our method is two sets of grid points where point correspondences across the two sets are known. The output is two hinge angles that give the lower and the upper bounds of the ARA for the two sets. Note that a part of this work was presented in [6].

2. Discrete rotation

It can appear strange that the Euclidean rotation used for the common task in geometric computation is problematic for many applications. Data are usually represented in the computer by integers or rational numbers. But the Euclidean rotation which requires sine and cosine functions is designed for real numbers. Therefore, the computation results given by a Euclidean rotation are, in most cases, represented by floating numbers which are approximations of real numbers. When an algorithm uses the Euclidean rotation for integer or rational data and then converts the floating values obtained by the algorithm into integer or rational numbers, precisions of these results may decrease. Another problem also arises for rotations in discrete space. It is well known that the Euclidean rotation is bijective and transitive. But when we convert the result obtained by the Euclidean rotation into a discrete space, it is easy to see that these two properties are lost [7,8]. The loss of these two properties leads to research on the discrete rotation.

There are two ways to compute a discrete rotation: using floating numbers and using only integers. The first way, in most cases, is easiest and allows us to use floating computation followed by the rounding function to obtain the set of grid points as the output. The main problem with the rounding function is the approximation due to the loss of the value after the decimal point. This approximation leads to lack of precision during computation. The second way does not have this problem, but avoiding floating numbers implies that sine and cosine functions should not be used. Computing rotations without trigonometrical functions requires development of a new method, which is a tough problem.

3. Hinge angles

We consider grid points in $\mathbb{Z}^2$ as the centers of pixels and rotate them in such a way that the rotation center has integer coordinates. Hinge angles are particular angles that make some points in $\mathbb{Z}^2$ rotated to points on the frontier between adjacent pixels. In this section, we give the definition of hinge angles and their properties related to Pythagorean angles.

3.1. Definition of hinge angles

Let $\boldsymbol{x} = (x, y)$ be a point in $\mathbb{R}^2$. We say that $\boldsymbol{x}$ has a semi-integer coordinate if $x + \frac{1}{2} \in \mathbb{Z}$ or $y + \frac{1}{2} \in \mathbb{Z}$. The set of points each of which has a semi-integer coordinate is called the half-grid and is denoted by $\mathscr{H}$. Thus, $\mathscr{H}$ represents the set of points on the frontiers of all pixels whose centroids are points in $\mathbb{Z}^2$.

Definition 1. An angle α is called a hinge angle if at least one point in $\mathbb{Z}^2$ exists such that its image by the Euclidean rotation with α belongs to $\mathscr{H}$.

Because $\mathscr{H}$ can be seen as the discontinuity of the rounding function, hinge angles can be regarded as the discontinuity of the DER. More simply, hinge angles determine a transit of a grid point from a pixel to its adjacent pixel during the rotation.

The following theorem is important because it shows that we can represent every hinge angle with three integers.

Theorem 2 (*Nouvel and Rémila [1]*). *An angle α is a hinge angle for a grid point $(P, Q) \in \mathbb{Z}^2$ if and only if there exists $K \in \mathbb{Z}$ such that*

$$2Q\cos\alpha + 2P\sin\alpha = 2K + 1. \tag{1}$$

Geometrically, a hinge angle α is formed by two rays going through (P, Q) and a half-grid point $(K + \frac{1}{2}, \lambda)$ where the two rays share the origin as their endpoints as shown in Fig. 1 (left). This theorem indicates that all calculations related to hinge angles can be done only with integers. Hereafter, α indicates a hinge angle.

We denote by $\alpha(P, Q, K)$ the hinge angle generated by an integer triplet (P, Q, K). Setting $\lambda = \sqrt{P^2 + Q^2 - (K + \frac{1}{2})^2}$, we easily derive the following equations from (1) and Fig. 1 (left),

$$\cos\alpha = \frac{P\lambda + Q(K + \frac{1}{2})}{P^2 + Q^2}, \quad \sin\alpha = \frac{P(K + \frac{1}{2}) - Q\lambda}{P^2 + Q^2}. \tag{2}$$

Note that we have a case where a half-grid point is $(\lambda, K + \frac{1}{2})$ instead of $(K + \frac{1}{2}, \lambda)$. In such a case, the above equations become

$$\cos\alpha = \frac{Q\lambda + P(K + \frac{1}{2})}{P^2 + Q^2}, \quad \sin\alpha = \frac{P\lambda - Q(K + \frac{1}{2})}{P^2 + Q^2}. \tag{3}$$

The symmetries on hinge angles are important because it allows us to restrict rotations in the first quadrant of the circle such that $\alpha \in [0, \pi/2]$.

Corollary 3. *Each triplet (P, Q, K) corresponds to four symmetrical hinge angles $\alpha + \pi k/2$ where $k = 0, 1, 2, 3$.*

Fig. 1 (right) gives an example of Corollary 3. In order to distinguish the case of $(K + \frac{1}{2}, \lambda)$ from that of $(\lambda, K + \frac{1}{2})$, we change the sign of K; we use $\alpha(P, Q, K)$ for the case of $(K + \frac{1}{2}, \lambda)$, and $\alpha(P, Q, -K)$ for the case of $(\lambda, K + \frac{1}{2})$. Because the symmetries allow us to restrict α to the range $[0, \pi/2]$, as mentioned above, we may assume that K is positive.

3.2. Properties related to Pythagorean angle

Because hinge angles are strongly related to Pythagorean angles, certain properties of Pythagorean angles are required to prove some properties of hinge angles. Thus, we first give the definition of Pythagorean angles and their properties.

Definition 4. An angle θ is called Pythagorean if and only if both its cosine and sine belong to the set of rational numbers $\mathbb{Q}$.

We can deduce from Definition 4 that a Pythagorean angle θ is represented by an integer triplet (a, b, c) such that

$$\cos\theta = \frac{a}{c}, \quad \sin\theta = \frac{b}{c}. \tag{4}$$