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Visual data denoising with a unified Schatten-p norm and ℓ_q norm regularized principal component pursuit

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ABSTRACT

To address the visual processing problem with corrupted data, in this paper, we propose a non-convex formulation to recover the authentic structure from the corrupted data. Typically, the specific structure is assumed to be low rank, which holds for a wide range of data, such as images and videos. Meanwhile, the corruption is assumed to be sparse. In the literature, such a problem is known as Robust Principal Component Analysis (RPCA), which usually recovers the low rank structure by approximating the rank function with a nuclear norm and penalizing the error by an ℓ_1-norm. Although RPCA is a convex formulation and can be solved effectively, the introduced norms are not tight approximations, which may cause the solution to deviate from the authentic one. Therefore, we consider here a non-convex relaxation, consisting of a Schatten-p norm and an ℓ_q-norm that promote low rank and sparsity respectively. We derive a proximal iteratively reweighted algorithm (PIRA) to solve the problem. Our algorithm is based on an alternating direction method of multipliers, where in each iteration we linearize the underlying objective function that allows us to have a closed form solution. We demonstrate that solutions produced by the linearized approximation always converge and have a tighter approximation than the convex counterpart. Experiments on benchmarks show encouraging results of our approach.

1. Introduction

The popularity of webcams and mobile phone cameras has generated a large amount of visual data. However, visual data are easily corrupted by artifacts arising from imaging devices or natural factors such as illumination. The human vision system could recognize the corruption with accumulated information and knowledge. However, it will result in irrelevant or noisy information in the computer vision community. Thus, a bunch of methods have been proposed to obtain authentic data for visual denoising tasks. Visual data denoising aims at reducing the noise from the observed visual documents [1–5]. Specifically, some approaches focus on statistical image modeling for the purpose of optimal signal representation and transmission, such as the Gaussian Scale Mixture (GSM) model, the variance-adaptive model or Bayesian estimation [6,5]. Portilla et al. presented a denoising method based on a local Gaussian scale mixture model in an overcomplete oriented pyramid representation [7]. The approaches mentioned above are based on the initial features of the visual data. Generally, better features will enhance the performance of representation. For instance, Shao et al. generated domain-adaptive global feature descriptors to obtain better performance in image classification [8]. Zhu et al. utilized weakly labeled data from other domains as the feature space for the visual categorization problem [9]. Based on a comprehensive feature space, some effective and promising denoising approaches are proposed by exploiting sparse and redundant representations over a trained dictionary [10]. Elad et al. proposed the K-SVD algorithm [11]. It was the first time that sparse modeling of image patches has been successfully applied in image denoising. The extension of K-SVD is proposed. Yan et al. exploited the sparsity within representation in the wavelet domain to handle high-level noises [12]. One reason for the success of the algorithm is the statistical properties of noise. It is natural to assume that the noise is sparse. Besides, the visual data such as images are probably of low rank structure [13]. For example, for a facial image taken under certain illumination conditions, the low-rank component captures the face, and the sparse component captures the light on the face [14]. Thus, the idea of turning the problem into a low rank matrix and a sparse matrix recovery problem has drawn considerable attention. This is the focus of our work. In the following, we first describe the problem.

1.1. The problem description

Suppose X is an observed data matrix in $R^{m \times n}$, where m is used to denote the ambient dimension of a sample and n is the number

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of samples. The problem can be formulated as

$$\min_{L,S} \operatorname{rank}(L)+\lambda\|S\|_0 \quad \text{s.t. } X=L+S, \tag{1}$$

where $L \in R^{m\times n}$ has a low rank structure that is assumed to be the authentic structure of the observed data and $S \in R^{m\times n}$ is assumed to be the sparse representation of the noise. Rank(L) is the rank of the matrix L, $\|S\|_0$ is the ℓ_0-norm which counts the number of non-zero entries in S, and λ is a parameter balancing the two components. The goal of the above optimization problem (1) is called Robust Principal Component Analysis (RPCA), aiming to recover the low-rank component L and sparse component S, with the constraint of $X=L+S$.

1.2. The reformulation and solutions

It is challenging to solve problem (1), because rank(L) and $\|S\|_0$ are both discontinuous and non-convex. In fact, it is NP-hard. A common strategy [15] is to relax the rank function to the convex nuclear norm $\|L\|_* = \sum_{i=1}^{\min(m,n)} \sigma_i(L)$, where σ_i denotes the ith singular value of L, and relax the ℓ_0-norm to the ℓ_1-norm $\|S\|_1 = \sum_{ij}|S_{ij}|$, where $|S_{ij}|$ is the magnitude of the (i,j) th element in S. Problem (1) can then be reformulated as

$$\min_{L,S}\|L\|_*+\lambda\|S\|_1 \quad \text{s.t. } X=L+S. \tag{2}$$

Candès et al. theoretically proved that if L and S satisfy certain assumptions, they can be recovered exactly via solving a convex program called Principal Component Pursuit with $\lambda = 1/\sqrt{\max\{m,n\}}$ [15]. Unlike the formulation defined in (1), RPCA in (2) is convex, and the optimal solution is tractable. An efficient solver for (2) is the Alternating Direction Method (ADM) [16] which guarantees to obtain the optimal solution. Another well-known first-order algorithm is the Accelerated Proximal Gradient (APG), which solves an unconstrained Stable Principal Component Pursuit (SPCP) problem [17] as follows:

$$\min_{L,S} \lambda_1\|L\|_*+\lambda_2\|S\|_1+\frac{1}{2}\|L+S-X\|_F^2, \tag{3}$$

where $\lambda_1>0$ and $\lambda_2>0$ are balancing parameters. APG is a fast method with a convergence rate $O(1/T^2)$, where T is the number of iterations.

1.3. Related works

As the RPCA model is capable of recovering the low rank component from grossly corrupted data and theoretical conditions to ensure the perfect recovery have been analyzed in depth, RPCA and its extensions have been applied to many applications, including background modeling [15], image alignment [18] and subspace segmentation [19]. Specifically, Hui et al. presented a patch-based algorithm using low-rank matrix recovery [20]. Wang et al. studied the problem of aligning correlated images by decomposing the matrix of corrupted images as the sum of a sparse matrix of errors and a low-rank matrix of recovered aligned images [21]. Hu et al. proposed a truncated nuclear norm regularization for estimating missing values from corrupted images [13].

There are several works aimed at improving the low-rank and sparse matrix recovery. Mu et al. [22] proposed an Accelerated RPCA using random projection. Zhou and Tao [23] developed a fast solver for low-rank and sparse matrix recovery with hard constraints on both L and S. To alleviate the challenges raised by coherent data, most recently, Liu et al. recovered the coherent data by Low-Rank Representation (LRR) [24,29]. Aybat et al. developed a fast first-order algorithm to solve the SPCP problem [25]. Fazel suggested to reformulating the rank optimization problem as a Semi-Definite Programming (SDP) problem [26]. An accelerated proximal gradient optimization technique was applied to solve the nuclear norm regularized least squares [27,28].

However, existing algorithms may lead to solutions that deviate from the original problem. Most previous works use the convex nuclear norm as a surrogate of the rank function and the ℓ_1-norm as a surrogate of the ℓ_0-norm, and then instead solve the new problem. But the nuclear norm is the sum of the singular values, while the rank function is the number of the non-zero singular values in which each singular value contributes equally. There are similar differences between the ℓ_0-norm and the ℓ_1-norm when performing a theoretical analysis [30]. Hence, the solution to the relaxed problem may be far from the original one. Some researchers instead consider non-convex surrogate functions.

The smoothed Schatten-p norm is a popular non-convex surrogate of the rank function defined as [31,32]

$$\begin{aligned}\ell_p(X) &= \operatorname{Tr}(X^TX+\epsilon I)^{p/2} \\ &= \sum_{i=1}^{n}(\sigma_i^2(X)+\epsilon)^{p/2}\end{aligned} \tag{4}$$

where I is the identity matrix with the same size as X, and $\ell_p(X)$ is differentiable for $p>0$ and nonconvex for $p<1$. Mohan and Fazel used the Schatten-p norm to replace the rank function and considered the problem [31]:

$$\begin{aligned}&\min \ell_p(X)\\ &\text{s.t.} A(X)=b,\end{aligned} \tag{5}$$

where $A: R^{m\times n} \longrightarrow R^p$ is a linear map, and $b \in R^p$ denotes the measurements. They also proposed the Iterative Reweighted Least Squares (IRLS) algorithm for rank minimization. Under certain conditions, IRLS-1 converges to the global minimum of the smoothed nuclear norm and IRLS-p converges to a stationary point of the corresponding non-convex yet smooth approximation to the rank function. Nie et al. [33] proposed the extended Schatten-p norm as an efficient surrogate of the rank function defined as

$$\begin{aligned}\ell_p &= \left(\sum_{i=1}^{\min(m,n)} \sigma_i^p\right)^{1/p}\\ &= \left(\operatorname{Tr}(X^TX)^{p/2}\right)^{1/p}.\end{aligned} \tag{6}$$

They derived an efficient algorithm to solve the above problem.

For the ℓ_0-norm, many non-convex surrogate functions have been proposed, e.g., ℓ_q-norm with $0<q<1$ [34], and Smoothly Clipped Absolute Deviation (SCAD) [35]. Nie et al. [32] used the Alternate Direction Method (ADM) to solve a similar problem for the non-convex matrix completion problem. Candés et al. [36] proposed an algorithm to solve the reweighted ℓ_1 minimization problem, which could better recover the ℓ_0-norm. The condition of sparse vector recovery has been given in [34].

The major drawback to the above approaches is that previous iteratively reweighted algorithms can only approximate either the low-rank component or the sparse one with a non-convex surrogate [37,38]. One important reason for this is that it is difficult to solve a problem whose objective function contains two or more nonsmooth terms. Thus, in this paper, we simultaneously approximate the low rank and sparse functions with non-convex surrogates.

1.4. Introducing our approach

In this paper, we propose a new formulation with the Schatten-p norm and ℓ_q-norm regularized Principal Component Pursuit (p,q-PCP) $(0<p,q<1)$ for recovering the low-rank and sparse matrices. We also provide an algorithm to solve such a non-convex problem with two non-smooth components. Empirically, our proposed Proximal Iteratively Reweighted Algorithm (PIRA) can solve p,q-PCP effectively without loss of efficiency. In each