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# Robust skeletonization using the discrete $\lambda$-medial axis

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ABSTRACT

Medial axes and skeletons are notoriously sensitive to contour irregularities. This lack of stability is a serious problem for applications in e.g. shape analysis and recognition. In 2005, Chazal and Lieutier introduced the $\lambda$-medial axis as a new concept for computing the medial axis of a shape subject to single parameter filtering. The $\lambda$-medial axis is stable under small shape perturbations, as proved by these authors. In this article, a discrete $\lambda$-medial axis (DLMA) is introduced and compared with the recently introduced integer medial axis (GIMA). We show that DLMA provides measurably better results than GIMA, with regard to stability and sensibility to rotations. We give efficient algorithms to compute the DLMA, and we also introduce a variant of the DLMA which may be computed in linear-time.


## 1. Introduction

The notion of medial axis has been introduced by Blum in the 60s (Blum, 1961, 1967). It has proved its usefulness in many practical applications, and numerous works were devoted to its properties and implementations. The original definition of the medial axis by Blum was based on a fire propagation analogy. However, its simplest definitions only require elementary geometry. In the continuous Euclidean space, the two following definitions can be used to formalise this notion: let $X$ be a bounded subset of $\mathbb{R}^n$;

- Interpretation (a) of the medial axis of $X$ consists of the centers of the $n$-dimensional balls that are included in $X$ but that are not included in any other $n$-dimensional ball included in $X$.
- Interpretation (b) of the medial axis of $X$ consists of the points $x \in X$ that have more than one nearest points on the boundary of $X$.

These two definitions differ only by a negligible set of points (see Matheron, 1988), in general interpretation (a) of the medial axis is a strict subset of interpretation (b). Notice that in some works, the term "skeleton" is used to refer to both interpretations, especially in the continuous framework. In this paper, we shall restrict the use of this term to the cases where the skeleton is topologically equivalent to the original shape.

To compute the medial axis approximately or exactly, different methods have been proposed, relying on different frameworks: discrete geometry (Borgefors et al., 1991; Ge and Fitzpatrick, 1996; Malandain and Fernández-Vidal, 1998; Rémy and Thiel, 2005; Hesselink and Roerdink, 2008), digital topology (Davies and Plummer, 1981; Vincent, 1991; Talbot and Vincent, 1992; Pudney, 1998), mathematical morphology (Serra, 1982; Soille, 1999), computational geometry (Attali and Lachaud, 2001; Ogniewicz and Kübler, 1995; Attali and Montanvert, 1996), partial differential equations (Siddiqi et al., 1999), and level-sets (Kimmel et al., 1995). In this paper, we focus on medial axes in the discrete grid $\mathbb{Z}^2$ or $\mathbb{Z}^3$, which are centered in the shape with respect to the Euclidean distance.

A major difficulty when using the medial axis in applications (e.g. shape recognition), is its sensitivity to small contour perturbations, in other words, its lack of stability. A recent survey (Attali et al., 2009) summarises selected relevant studies dealing with this topic. This difficulty can be expressed mathematically: the transformation which associates a shape to its medial axis is only semi-continuous. This fact, among others, explains why it is usually necessary to add a filtering step (or pruning step) to any method that aims at computing the medial axis.

Hence, there is a rich literature devoted to medial axis pruning, in which different criteria were proposed in order to discard "spurious" medial axis points or branches: see Attali et al. (1995), Ogniewicz and Kübler (1995), Attali and Montanvert (1996), Malandain and Fernández-Vidal (1998), Attali and Lachaud (2001), Svensson and Sanniti di Baja (2003), Bai et al. (2007), Couprie et al. (2007), Hesselink and Roerdink (2008), to cite only a few. However, we lack theoretical justification, that is, a formalised argument that would help to understand why a filtering criterion is better than another.

In 2005, Chazal and Lieutier introduced the $\lambda$-medial axis and studied its properties, in particular those related to stability

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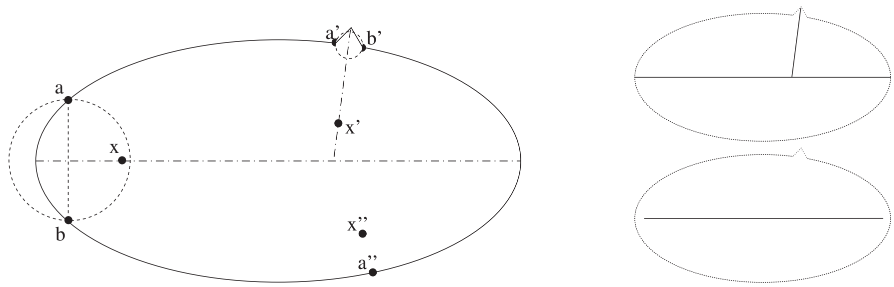


**Fig. 1.** Illustration of the $\lambda$-medial axis. Left: Points $x$, $x'$ and $x''$ and their respective closest boundary points. Top right: $\lambda$-medial axis with $\lambda = \epsilon$, a very small positive real number. Bottom right: $\lambda$-medial axis with $\lambda = d(a',b')/2 + \epsilon$.

(Chazal and Lieutier, 2005). Consider a bounded subset $X$ of $\mathbb{R}^n$, as for example, for $n = 2$, the region enclosed by the solid curve depicted in Fig. 1 (left). Let $x$ be a point in $X$, we denote by $\Pi(x)$ the set of points of the boundary of $X$ that are closest to $x$. For example in Fig. 1, we have $\Pi(x) = \{a,b\}$, $\Pi(x') = \{a',b'\}$ and $\Pi(x'') = \{a''\}$. Let $\lambda$ be a non-negative real number, the $\lambda$-medial axis of $X$ is the set of points $x$ of $X$ such that the smallest $n$-dimensional ball[1] including $\Pi(x)$ has a radius greater than or equal to $\lambda$. Notice that the 0-medial axis of $X$ is equal to $X$, and that any $\lambda$-medial axis with $\lambda > 0$ is included in the medial axis according to definition (b). We show in Fig. 1 (right) two $\lambda$-medial axes with different values of $\lambda$.

A major outcome of Chazal and Lieutier (2005) is the following property: informally, for "regular" values of $\lambda$, the $\lambda$-medial axis remains stable under perturbations of $\overline{X}$ that are small with respect to the Hausdorff distance. Typical non-regular values are radii of locally largest maximal $n$-dimensional balls.

This property is a strong argument in favor of the $\lambda$-medial axis, especially in the absence of such result for other proposed criteria.

In the field of computational geometry, the $\lambda$-medial axis has been exploited in particular by Samozino et al. (2006) to propose a robust method for reconstructing surfaces from point clouds. Also, notions closely related to the $\lambda$-medial axis have led Chazal et al. to propose stable approximations of tangent planes and normal cones from noisy samples (Chazal et al., 2009).

In the discrete grids, namely $\mathbb{Z}^2$ and $\mathbb{Z}^3$, a similar filtering criterion has been considered in independent works (Malandain and Fernández-Vidal, 1998; Hesselink and Roerdink, 2008). It consists of selecting for each medial axis point two of its closest boundary points, and using the distance between these two points as filtering criterion. The work of Hesselink and Roerdink (2008) provides a linear-time algorithm to compute a filtered medial axis based on this criterion, which exhibits good noise robustness properties in practice.

In this article (which extends Chaussard et al. (2009), a preliminary version published in the DGCI conference proceedings), we introduce the definition of a discrete $\lambda$-medial axis (DLMA) in $\mathbb{Z}^n$. We evaluate experimentally its stability and rotation invariance in 2D and 3D. In this experimental study, we compare it with the previously introduced integer medial axis (GIMA) (Hesselink et al., 2005; Hesselink and Roerdink, 2008) and show that the DLMA provides measurably better results. Furthermore, we introduce a variant of the DLMA which may be computed in linear time, for which the results are very close to those of the DLMA, and which is only slightly slower than the one proposed by Hesselink and Roerdink (2008).

[1] The center of this ball is also the projection of $x$ onto the convex hull of $\Pi(x)$ (see Chazal et al., 2009).

## 2. The $\lambda$-medial axis

Let us first recall the original definition of the $\lambda$-medial axis given by Chazal and Lieutier.

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, the Euclidean distance between $x$ and $y$ is denoted by $d(x,y)$, in other terms, $d(x,y) = (\sum_{k=1}^{n}(y_k - x_k)^2)^{\frac{1}{2}}$.

Let $X$ be a finite subset of $\mathbb{R}^n$ or $\mathbb{Z}^n$. We set $d(y,X) = \min_{x \in X}\{d(y,x)\}$. We denote by $|X|$ the number of elements of $X$, and by $\overline{X}$ the complement of $X$.

Let $E$ be either $\mathbb{R}^n$ or $\mathbb{Z}^n$. Let $x \in E$, $r \in \mathbb{R}^+$, we denote by $B_r(x)$ the *n-dimensional ball* (or *ball* for simplicity) *of radius r centered on x*, defined by $B_r(x) = \{y \in E | d(x,y) \leqslant r\}$. We also define $\mathring{B}_r(x) = \{y \in E \mid d(x,y) < r\}$.

Let $X \subseteq E$. A ball $B_r(x) \subseteq X$ is *maximal for X* if it is not strictly included in any other ball included in $X$. The *Euclidean Medial Axis of X*, denoted by EMA($X$), is the set of the centers of all the maximal balls for $X$.

Let $S$ be a non-empty subset of $E$, and let $x \in E$. The *projection of x on S*, denoted by $\Pi_S(x)$, is the set of points $y$ of $S$ which are at minimal distance from $x$; more precisely,

$$\Pi_S(x) = \{y \in S | \forall z \in S, d(y,x) \leqslant d(z,x)\}.$$

If $X$ is a subset of $E$, the *projection of X on S* is defined by $\Pi_S(X) = \bigcup_{x \in X} \Pi_S(x)$.

Let $S \subset \mathbb{R}^n$, we denote by $R(S)$ the radius of the smallest ball enclosing $S$, that is, $R(S) = \min\{r \in \mathbb{R} | \exists y \in \mathbb{R}^n, B_r(y) \supseteq S\}$.

The $\lambda$-medial axis may now be defined based on these notions.

**Definition 1** (Chazal and Lieutier, 2005). Let $X$ be an open bounded subset of $\mathbb{R}^n$, and let $\lambda \in \mathbb{R}^+$. The *$\lambda$-medial axis of X* is the set of points $x$ in $X$ such that $R(\Pi_{\overline{X}}(x)) \geqslant \lambda$.

## 3. A discrete $\lambda$-medial axis

Transposing the original definition of the $\lambda$-medial axis directly to $\mathbb{Z}^n$ would lead to an unsatisfactory result. For instance, consider a horizontal ribbon in $\mathbb{Z}^2$ with constant, even width and infinite length. Clearly, the projection of any point of this set on its complementary set is reduced to a singleton. This is why, if we keep the same definition, any $\lambda$-medial axis of this object with $\lambda > 0$ would be empty.

In order to avoid such unwanted behaviour, we replace the projection by the so-called extended projection (Couprie et al., 2007). The extended projection was originally introduced in order to propose a discrete definition of the bisector function, another indicator used to filter skeletons.