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An introduction to simple sets

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ABSTRACT

Preserving topological properties of objects during thinning procedures is an important issue in the field of image analysis. In this context, we present an introductory study of the new notion of *simple set* which extends the classical notion of simple point. Similarly to simple points, simple sets have the property that the homotopy type of the object in which they lie is not changed when such sets are removed. Simple sets are studied in the framework of cubical complexes which enables, in particular, to model the topology in $\mathbb{Z}^n$. The main contributions of this article are: a justification of the study of simple sets (motivated by the limitations of simple points); a definition of simple sets and of a subfamily of them called *minimal simple sets*; the presentation of general properties of (minimal) simple sets in *n*-D spaces, and of more specific properties related to "small dimensions" (these properties being devoted to be further involved in studies of simple sets in 2, 3 and 4-D spaces).

1. Introduction

Topological properties are fundamental in many applications of image analysis, in particular in research fields where the retrieval and/or preservation of topology of real complex structures is required.

Topology-preserving operators, like homotopic skeletonisation, are used to transform an object while leaving unchanged its topological characteristics. In discrete grids ($\mathbb{Z}^2, \mathbb{Z}^3$, or $\mathbb{Z}^4$), such transformations can be defined and efficiently implemented thanks to the notion of *simple point* (Kong and Rosenfeld, 1989; Bertrand, 1994; Couprie and Bertrand, 2009): intuitively, a point of an object is called simple if it can be deleted from this object without altering its topology. A typical topology-preserving transformation based on simple points deletion, that we call *guided homotopic thinning* (Davies and Plummer, 1981; Couprie et al., 2007), may be described as follows. The input data consists of a set X of points in the grid (called object), and a subset $K \subset X$ (called constraint set). Let $X_0 = X$. At each iteration i, choose a simple point $x_i \in X_i \setminus K$ according to some criterion (*e.g.*, a priority function) and set $X_{i+1} = X_i \setminus \{x_i\}$. Continue until reaching a step n such that no simple point for X_n remains in $X_n \setminus K$. We call the result of this process a *homotopic skeleton of X constrained by K*. Notice that, since several points may have the same priority, there may exist several homotopic skeletons for a given pair (X, K).

The most common example of priority function for the choice of x_i is a distance map which associates to each point of X its distance from the boundary of X. In this case, the points which are closest to the boundary are chosen first, resulting in a skeleton which is "centered" in the original object. In some particular applications, the priority function may be obtained through a grey-scale image, for example when the goal is to segment objects in this image while respecting topological constraints (Dokládal et al., 1999). In the latter case, the order in which points are considered does not rely on geometrical properties, and may be affected by noise.

One drawback of thinning algorithms that work in the manner we have described is that the final set X_n is not always minimal (Passat et al., 2005). The problem here is that even though X_n contains no simple point outside the constraint set K, it is still possible for $X_n \setminus K$ to include non-empty subsets D which have the property that X_n can be "deformed", in a sense that will be made precise in Definition 7, onto the smaller set $X_n \setminus D$ (so that X_n is "homotopy equivalent" in a discrete sense to $X_n \setminus D$). A subset D that has this property will be called a *simple set* (for X_n). An example of such a set is depicted in Fig. 1; if X_n is the 3-D set shown in that figure, then the set $D \subset X_n$ (in light grey) is simple for X_n.

One way to address this problem would be to try to further reduce the set X_n by finding and deleting some subset D of $X_n \setminus K$ that is simple for X_n. To put this idea into practice, we need good ways of finding sets in $X_n \setminus K$ that are simple for X_n.

We are, in particular, interested by simple sets which are *minimal*, in the sense that they do not strictly include any other simple set, since it is sufficient to detect such sets in order to carry on thinning. Also, we can hope that such *minimal simple sets* (*i*) have a specific structure which could make them easier to analyse, and (*ii*) are sufficient to deal with the whole problem of simple set removal.

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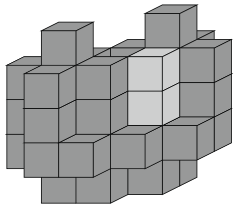

Fig. 1. A set X_n composed of 32 points (considered in 26-adjacency, see also Figs. 3a and 4a), which does not contain any simple point, but which includes a subset D (in light grey), whose removal from X_n provides a set $X_n \setminus D$ "homotopy equivalent" in a discrete sense to X_n. In particular, it is possible to further reduce $X_n \setminus D$ to a single point by iterative removal of simple points.

The sequel of this article is organised as follows. In Section 2, we propose a discussion on "topological artifacts" which may appear in discrete images, especially during topology-preserving reduction procedures (generally based on simple points). This discussion leads to define the notion of simple set which provides a way to "break" some of these artifacts, and then enables to improve the efficiency of reduction procedures. In Section 3, the framework of cubical complexes is described. Indeed, we propose to study simple sets in this framework, from which we can retrieve the classical notions of digital topology in $\mathbb{Z}^n$, but which also enables to deal with more general categories of cubical objects. Section 4 presents the main notions of topology preservation in cubical spaces and formally introduces the definitions of simple sets and minimal simple sets. General properties of such sets (valid in any dimension) are proposed and proved in Section 5, while more specific ones, devoted to "low dimensions" are proposed and proved in Section 6. Discussions and perspectives regarding further works on simple sets are provided in Section 7.

2. Why are simple sets useful?

2.1. Topological artifacts: the notion of lump

Let us consider the guided homotopic thinning procedure described in Section 1. When performing such a procedure, the result is expected to fulfil a property of minimality. This is indeed the case since the result X_n is minimal in the sense that it contains no simple point outside of K. However, we could formulate a stronger minimality requirement, which seems natural for this kind of transformation: informally, the result X_n should not strictly include any subset Y which is "topologically equivalent" to X, and such that $K \subseteq Y \subset X_n$. We say that a homotopic skeleton of X constrained by K is *globally minimal* if it fulfils this condition.

Now, a fundamental question arises: is any homotopic skeleton globally minimal? Let us illustrate this problem in dimensions 2 and 3. In $\mathbb{Z}^2$, consider a full rectangle X of any size, and the constraint set $K = \emptyset$. Obviously, this object X is topologically equivalent to a single point, thus only homotopic skeletons which are singletons are globally minimal. Rosenfeld (1970) proved that any homotopic skeleton of X is indeed reduced to a single point.

However, in dimensions $n \geqslant 3$, this property does not hold: if X is *e.g.* a full $k \times k \times k$ cube ($k \geqslant 5$), we may find a homotopic skeleton of X, with empty constraint set, which is not reduced to a single point (see Fig. 1). A classical counter-example is the Bing's house with two rooms (Bing, 1964), illustrated in Fig. 2. One can enter the lower room of the house by the chimney passing through the upper room, and *vice versa*. A discrete version X_1 of the Bing's house is displayed in Fig. 5a. It can be seen that the Bing's house can be carved from a full cube by iterative removal of simple points. It can also be seen that X_1 contains no simple point: deleting any point from X_1 would create a "tunnel".

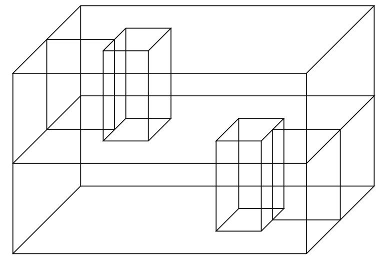

Fig. 2. A Bing's house with two rooms visualised as a 2-D surface in $\mathbb{R}^3$ (see text).

It could be argued that objects like Bing's houses are unlikely to appear while processing real (noisy) images, because of their complex shape and their size. However, Passat et al. (2007) found that there exists a large class of objects (of any topology) presenting similar properties, some of them being quite small (see Figs. 3 and 4). Such objects will be called *lumps* and can be defined, as follows, thanks to the notion of simple-equivalence.

Definition 1. Let $n \geqslant 1$. Let $X, X' \subset \mathbb{Z}^n$. We say that X and X' are *simple-equivalent* if there exists a sequence of sets $\langle X_i \rangle_{i=0}^{t} (t \geqslant 0)$ such that $X_0 = X, X_t = X'$, and for all $i \in [1, t]$, we have either:

(i) $X_i = X_{i-1} \setminus \{x_i\}$, where $x_i \in X_{i-1}$ is a simple point for X_{i-1}; or
(ii) $X_{i-1} = X_i \setminus \{x_i\}$, where $x_i \in X_i$ is a simple point for X_i.

Definition 2. Let $n \geqslant 1$. Let $X' \subset X \subset \mathbb{Z}^n$ such that X and X' are simple-equivalent. If X does not contain any simple point outside X', then we say that X is a *lump relative to* X', or simply a *lump*.

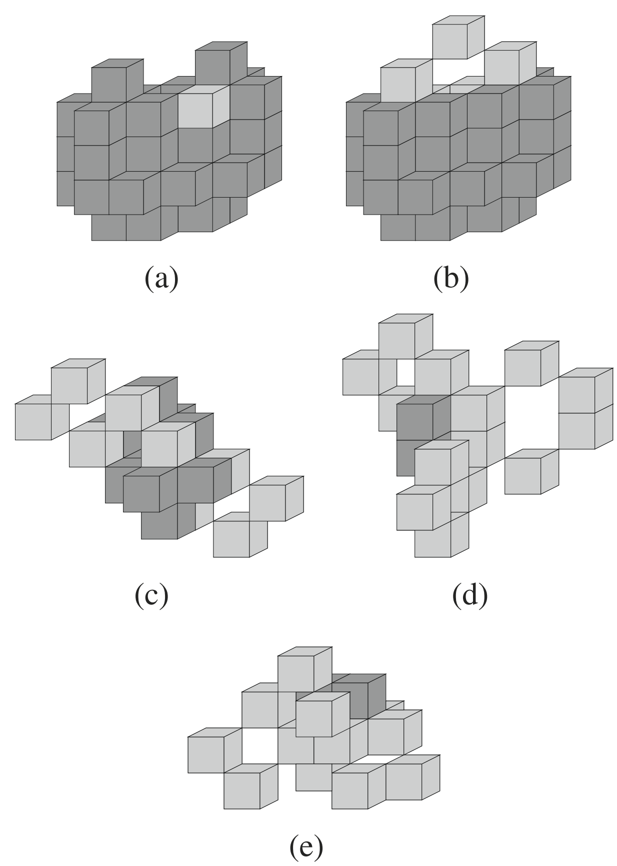

Fig. 3. Examples of 3-D lumps L_i ((a–e): L_1 to L_5), considered in 26-adjacency. Topologically equivalent subsets are depicted in light grey. The set L_i ($i = 1$ to 5) has $i - 1$ tunnel(s).