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Relationship between restricted dissimilarity functions, restricted equivalence functions and normal E_N-functions: Image thresholding invariant

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Abstract

In this paper, we present the definition of restricted dissimilarity function. This definition arises from the concepts of dissimilarity and equivalence function. We analyze the relation there is between restricted dissimilarity functions, restricted equivalence functions (see [Bustince, H., Barrenechea, E., Pagola, M., 2006. Restricted equivalence functions. Fuzzy Sets Syst. 157, 2333–2346]) and normal E_N-functions. We present characterization theorems from implication operators and automorphisms. Next, by aggregating restricted dissimilarity functions in a special way, we construct distance measures of Liu, proximity measures of Fan et al. and fuzzy entropies. We also study diverse interrelations between the above-mentioned concepts. These interrelations enable us to prove that under certain conditions, the threshold of an image calculated with the algorithm of Huang and Wang [Huang, L.K., Wang, M.J., 1995. Image thresholding by minimizing the measure of fuzziness. Pattern Recognit. 28 (1), 41–51], with the methods of Forero [Forero, M.G., 2003. Fuzzy thresholding and histogram analysis. In: Nachtegael, M., Van der Weken, D., Van de Ville, D., Kerre, E.E. (Eds.), Fuzzy Filters for Image Processing. Springer, pp. 129–152] or with the algorithms developed in [Bustince, H., Barrenechea, E., Pagola, M., 2007. Image thresholding using restricted equivalence functions and maximizing the measures of similarity. Fuzzy Sets Syst. 158, 496–516] is always the same, that is, it remains invariant.

Keywords: Restricted equivalence functions; Restricted dissimilarity functions; Distance measures; Fuzzy entropy; Implication operators; Image thresholding

1. Introduction

We know that the segmentation of digital images is the process of dividing an image into disjoint parts, regions or classes. Each one of these classes represents an object of the image.

A widely used method for segmenting images is called *gray-level image segmentation method*. With this method, in images composed of an object and the background, the selection of pixels that belong to the object and the selection of those that belong to the background is done by establishing a threshold t from which the pixels with the highest intensities belong to the background (or object) and with lowest intensities belong to the object (or background).

In the literature there are many methods for calculating the threshold t of an image (e.g., Forero and Rojas, 2000; Forero et al., 2001; Glasbey, 1993; Jan et al., 1997; Parker, 1997; Pratt, 1991; Sankur and Sezgin, 2004; Sahoo et al., 1988). Nevertheless (see Bustince et al., 2007), considering that fuzzy set theory Zadeh (1965) has worked well in the treatment of models that present ambiguity and highly noisy data, this theory is an interesting alternative for

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determining the best threshold, in order to obtain a good segmentation of the image considered (see Bustince et al., 2007; Chi et al., 1998; Jan et al., 1997; Lin and Lee, 1996; Pal and Pal, 1993). Within the framework of this theory the most popular algorithms are those that use the concept of fuzzy entropy (Bezdek et al., 1999; Forero, 2003; Huang and Wang, 1995; Tizhoosh, 2005). The best known algorithm is the following:

Algorithm 1

(a) Assign L fuzzy sets Q_t to each image Q. Each one is associated to a level of intensity t ($t = 0, 1, \ldots, L-1$), of the grayscale L used.
(b) Calculate the entropy of each one of the L fuzzy sets Q_t associated with Q.
(c) Take as the *best threshold* the gray level t associated with the fuzzy set corresponding to the lowest entropy. (The justification for this choice is explained in (Forero, 2003; Huang and Wang, 1995).)

The objective of this paper is to prove that:
If in Algorithm 1, we use:

(1) Fuzzy entropies generated from normal E_N-functions or
(2) Fuzzy distances generated from restricted dissimilarity functions or
(3) Fuzzy similarities constructed from restricted equivalence functions (see Bustince et al., 2006),

then the threshold t calculated is the same in the three cases; that is, the threshold remains invariant.

This objective has led us to study the relation there is between Liu's distance measures (see Liu, 1992), the proximity measures of Fan and Xie (1999) and Fan et al. (1999) and fuzzy entropies (see De Luca and Termini, 1972). In order to do so we are first going to define and study the concepts of restricted dissimilarity functions, restricted equivalence functions (see Bustince et al., 2006) and normal E_N-functions and next we are going to analyze the interrelations between all of them. These interrelations will enable us to achieve the objective described above.

This paper is organized as follows: In Section 2, we recall the basic notions of Fuzzy Set Theory that we will use. In Section 3, we present and justify the definition of *restricted dissimilarity functions*. Next we show a construction method from two automorphisms. Then we characterize our functions using implication operators and automorphisms. In Section 4, we present a method for the construction of Liu's distance measures (see Liu, 1992) and proximity measures of Fan and Xie (1999) and Fan et al. (1999). In the construction method, we study restricted dissimilarity functions are aggregated by means of aggregation operators that satisfy special properties. In Section 5, we study the conditions that we must demand from the aggregation operators in order to obtain different interrelations between Liu's distance measures and similarity measures. In Section 6, we present the notion of normal E_N-functions and two construction methods. In Section 7, we analyze the properties that we must demand from aggregation operators in order to construct fuzzy entropies from E_N-functions. In Section 8, we present the relation between similarity measures, Liu's distance measures and De Luca and Termini's entropy. In Section 9, we justify the study carried out in this paper by applying the results obtained regarding the interrelations between the concepts of Liu's distance measures, similarity measures and fuzzy entropies generated from normal E_N-functions. That is, we present the invariance of the threshold calculated with Algorithm 1 as a consequence of this interrelation.

2. Preliminaries

All of the results we present in this section will be used throughout the paper.

2.1. Fuzzy negations

Let $N : [0,1] \to [0,1]$. N is a *fuzzy negation*, iff:

(1) $N(0) = 1$ and $N(1) = 0$,
(2) $N(x) \leqslant N(y)$, if $x \geqslant y$ (monotonicity).A fuzzy negation is *strict*, iff,
(3) $N(x)$ is *continuous*,
(4) $N(x) < N(y)$, for $x > y$ for all $x, y \in [0,1]$.
A fuzzy negation is *involutive*, iff
(5) $N(N(x)) = x$, for all $x \in [0,1]$.

In this paper, we shall use only *strong negations*; that is, fuzzy negations that are strict and therefore they are involutive. In (Klir and Folger, 1988), it is proven that every strong negation has a single equilibrium point e, i.e. there is a single point $e \in (0,1)$ such that $N(e) = e$.

The main representation theorem for strong negations was obtained by Trillas (1979). We cite below that result in a slightly modified form which is more suitable in the sequel (see Ovchinnikov and Roubens, 1991). First we need the definition of an *automorphism* of a real interval $[a,b] \subset \mathbb{R}$ (see Fodor and Roubens, 1994). This notion will be used extensively throughout the paper, especially in the case of the unit interval $[0,1]$.

Definition 1. A continuous, strictly increasing function $\varphi : [a,b] \to [a,b]$ with boundary conditions $\varphi(a) = a$, $\varphi(b) = b$ is called an *automorphism* of the interval $[a,b] \subset \mathbb{R}$.

Theorem 1. *A function $N : [0,1] \to [0,1]$ is a strong negation if and only if there exists an automorphism φ of the unit interval such that $N(x) = \varphi^{-1}(1 - \varphi(x))$.*

We will denote with $\mathscr{F}(U)$ the set of all the fuzzy sets defined on the finite referential and not empty set U