Pattern Recognition Letters 34 (2013) 1589–1601

Contents lists available at SciVerse ScienceDirect

Pattern Recognition Letters

journal homepage: www.elsevier.com/locate/patrec

Quantale-based autoassociative memories with an application to the storage of color images ☆

Marcos Eduardo Valle [a,*], Peter Sussner [b,1]

[a] Department of Mathematics, University of Londrina, CEP 86051-990 Londrina, PR, Brazil
[b] Department of Applied Mathematics, University of Campinas, CEP 13081-970 Campinas, SP, Brazil

ARTICLE INFO

Article history:
Available online 10 April 2013

Communicated by Manuel Graña

Keywords:
Lattice computing
Complete lattice
Quantale
Morphological neural network
Neural associative memory
Color image

ABSTRACT

In recent years, lattice computing has emerged as a new paradigm for processing lattice ordered data such as intervals, Type-1 and Type-2 fuzzy sets, vectors, images, symbols, graphs, etc. Here, the word "lattice" refers to a mathematical structure that is defined as a special type of a partially ordered set (poset). In particular, a complete lattice is a poset that contains the infimum as well as the supremum of each of its subsets. In this paper, we introduce the quantale-based associative memory (QAM), where the notion of a quantale is defined as a complete lattice together with a binary operation that commutes with the supremum operator. We show that QAMs can be effectively used for the storage and the recall of color images.

1. Introduction

To the best of our knowledge, most associative memory (AM) models for storing and retrieving color images represent adaptations of AMs for real-valued patterns such as the following two straightforward approaches.

The first approach relies on the RGB system in which a color element c is expressed in terms of red, green, and blue components, i.e., $c = [c_r, c_g, c_b]$ (Acharya and Ray, 2005; Gonzalez and Woods, 2002). Hence, a color image can be decomposed into three gray-scale images that are usually stored in three separate gray-scale AMs. Zheng et al. (2010) applied this approach to color images using a class of Cohen–Grossberg networks. In fact, any gray-scale AM model can be extended to cope with color patterns using this approach, however the lack of interaction between the color channels may affect the noise tolerance of the resulting color AM.

The second approach used by Vazquez and Sossa (2008, 2009) is based on the 24-bit representation of digital color images. Precisely, an integer q from 0 to $2^{24} - 1$ is mapped to a color value using the equation $q = 256^2 c_r + 256 c_b + c_g$, where c_r, c_b, c_g in $\{0, 1, \ldots, 255\}$ denote the red, green, and blue components of the color element in the digital 8-bit RGB system. In view of this representation, any AM for integer- or real-valued patterns can generate associations between color images. A drawback of this approach is that visually very different color elements may be associated with similar integer values. For example, the integers 255 and 256 are respectively assigned to the pure blue and visually black elements whose RGB representations are $[0, 0, 255]$ and $[0, 1, 0]$.

Recall that the RGB model represents the most popular color space for storing, processing, and displaying color images. However, this model is not suited to quantify the perceptual difference between images (Plataniotis et al., 1999). In fact, the Euclidean distance between two color elements in the RGB system may not reflect the color difference perceived by the human eye. For instance, at higher illumination, the eye is more sensitive if the color has not been saturated (Acharya and Ray, 2005). Therefore, the aforementioned approaches are not recommended in application areas such as multimedia, telecommunications, and printing industry, where the perceptual quality of the restored image is very important.

In contrast to the approaches presented above, *sparsely connected autoassociative morphological memories* (SCAMMs) were not derived from gray-scale models (Valle, 2009). These models are also known as *sparsely connected lattice autoassociative memories* (Valle and Grande Vicente, 2012) and can be defined on any complete lattice, a mathematical structure that is defined in terms of a partial ordering on a set (Birkhoff, 1993). In particular, the set of colors in the CIELab system can be equipped with a partial ordering scheme that depends on the distance with respect to a certain color reference (Valle and Grande Vicente, 2011). Since CIELab is a perceptually uniform color space in the sense that the

☆ This work was supported in part by CNPq under Grant Nos. 304240/2011-7 and 309608/2009-0, by FAPESP under Grant No. 2011/10014-3, and by Fundação Araucária.

* Corresponding author. Tel.: +55 043 3371 4961.
E-mail addresses: valle@uel.br (M.E. Valle), sussner@ime.unicamp.br (P. Sussner).
[1] Tel.: +55 019 3521 5959

0167-8655/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.patrec.2013.03.034

Euclidean distance between two color points corresponds to the perceptual difference by the human visual system (Acharya and Ray, 2005), CIELab-based SCAMMs can be used in applications that emphasize the perceptual image quality. Moreover, SCAMMs represent attractive models for storing and reconstructing multi-valued, large-scale patterns such as color images since the following factors contribute to a very low computational effort:

1. Construction of the weight matrix using only kn^2 comparisons;
2. Sparsity of the weight matrix;
3. Recall phase requiring only calculations of suprema and infima (i.e., in most cases, maxima and minima).

Recently, Valle and Grande Vicente proved that the weight matrices of SCAMMs arise from the ones of classical *morphological associative memory* (MAM) models via the application of a thresholding operation (Valle and Grande Vicente, 2012). Recall that – in comparison to SCAMMs – classical MAMs are defined in a richer mathematical structure (than merely a complete lattice) that includes a group operation apart from the lattice operations (''meet'' and ''join'') (Birkhoff, 1993; Sussner and Valle, 2006; Sussner and Esmi, 2011; Ritter et al., 1998). We believe that this is the reason why MAMs exhibit a better error correction capability than SCAMMs in certain applications concerning the storage and recall of gray-scale images (Valle, 2009). However, the MAM models, whose weight matrices are fully connected, are computationally too expensive for dealing with large-scale patterns.

These considerations motivated us to introduce an AM model on a quantale. In applications using color images, a quantale arises by endowing the CIELab system in spherical coordinates with a binary operation that is associative and has an identity. In this paper, we prove that the resulting quantale-based autoassociative memory (QAM) exhibits optimal absolute storage capacity. Moreover, we show by means of theoretical and experimental results that QAMs have an improved tolerance with respect to noise in comparison to SCAMMs. Like the sparsely connected autoassociative fuzzy implicative memories and other traditional sparsely connected associative memory models (Bohland and Minai, 2001; Valle, 2010), the QAM model can be organized in a small-world network.

This paper is organized as follows: Section 2 discusses the mathematical background on the notion of a *quantale* that consists of a complete lattice with an associative binary operation. Section 3 presents a class of AM models for computing in quantales. Section 4 reviews both the RGB and CIELab color models and introduces the spherical CIELab quantale. The *spherical CIELab quantale-based autoassociative memories* which can be used for the storage of color images are presented in Section 5. The paper finishes with the concluding remarks in Section 7.

2. Mathematical background

The *quantale-based autoassociative memory (QAM)* that we introduce in this paper represents an approach towards *computational intelligence based on lattice theory* (Kaburlasos and Ritter, 2007). We could also refer to the QAM as a *lattice computing* model if the technical term ''lattice computing'' is interpreted in a wider sense than originally envisaged (Graña, 2008). Informally speaking, lattice computing should refer to a collection of techniques and methodologies for analyzing and processing data by means of operations in mathematical lattices (Birkhoff, 1993). Many lattice computing approaches towards computational intelligence such as fuzzy lattice neurocomputing and reasoning techniques (Kaburlasos et al., 2007; Kaburlasos and Kehagias, 2006; Kaburlasos and Petridis, 2000; Li et al., 2012; Liu et al., 2011) as well as classical and fuzzy morphological associative memories (Sussner and Valle, 2006; Graña et al., 2009; Ritter and Urcid, 2011; Valle and Sussner, 2011; Ritter et al., 1998; Valle and Sussner, 2008) and other morphological neural network models (Sussner and Esmi, 2011; Pessoa and Maragos, 2000; Ritter and Sussner, 1996) require an additional algebraic structure apart from a complete lattice structure. The QAM model represents yet another addition to the list of these approaches towards computational intelligence.

The lattice computing model introduced in this paper is defined in an algebraic structure called *(unital) quantale*. The concept of a quantale was devised by Mulvey to provide an appropriate framework for the logic of quantum mechanics (Mulvey, 1986) and is also closely related to the linear logic of Girard (1987), Yetter (1990), the notion of residuated lattices (Ward and Dilworth, 1939; Russo, 2010), and the algebraic structure called *complete lattice-ordered double monoid* (clodum) introduced by Maragos (2005). The following subsection provides some mathematical background on quantales.

2.1. Quantales

A *quantale* $\mathcal{Q}$ is a complete lattice $\mathcal{Q}$ together with an associative binary operation called *multiplication* which distributes on both sides over arbitrary suprema. Recall that a partially ordered set $\mathcal{Q}$ is a complete lattice if every non-empty subset of $\mathcal{Q}$ has an infimum and a supremum in $\mathcal{Q}$. We denote the supremum and the infimum of a non-empty set $X \subseteq \mathcal{Q}$ by $\bigvee X$ and $\bigwedge X$, respectively. In particular, $\bigvee_{j=1}^n x_j$ and $\bigwedge_{j=1}^n x_j$ are respectively used to represent the supremum (or maximum) and the infimum (or minimum) of a finite set $X = \{x_1, \ldots, x_n\} \subseteq \mathcal{Q}$. Let the symbol "$\cdot$" denote a binary operation $\mathcal{Q} \times \mathcal{Q} \to \mathcal{Q}$. The operation $\cdot$ is distributive over arbitrary suprema if the following equations hold true for every $q \in \mathcal{Q}$ and for all non-empty set $X \subseteq \mathcal{Q}$:

$$q \cdot \left(\bigvee X\right) = \bigvee_{x \in X} \{q \cdot x\} \text{ and } \left(\bigvee X\right) \cdot q = \bigvee_{x \in X} \{x \cdot q\}. \tag{1}$$

Eq. (1) implies that the multiplication is increasing in both arguments, i.e., if $x \leqslant y$ then both inequalities $x \cdot q \leqslant y \cdot q$ and $q \cdot x \leqslant q \cdot y$ hold true for all $q \in \mathcal{Q}$. Moreover, the least element of $\mathcal{Q}$, denoted by the symbol "$\perp$", is a zero or absorbing element of the quantale $\mathcal{Q}$, i.e., $q \cdot \perp = \perp \cdot q = \perp$ for all $q \in \mathcal{Q}$ (Russo, 2010).

We speak of a *commutative quantale* if the multiplication is commutative. Similarly, we speak of a *unital quantale* if the multiplication has an identity or neutral element, i.e., if there exists $e \in \mathcal{Q}$ such that $e \cdot x = x \cdot e = x$ for all $x \in \mathcal{Q}$. The multiplication of a quantale $\mathcal{Q}$ is always residuated (Russo, 2010; Blyth and Janowitz, 1972; Blyth, 2005). Specifically, there exist binary operations $\backslash$ and $/$ in $\mathcal{Q}$ such that the following equivalences hold true for $x, y, z \in \mathcal{Q}$:

$$x \cdot y \leqslant z \quad \text{iff} \quad x \leqslant z/y \quad \text{iff} \quad y \leqslant x \backslash z. \tag{2}$$

The operations $/$ and $\backslash$ are respectively called the right and left *residuals* or *divisions* of $\cdot$. For any $x, y \in \mathcal{Q}$, these two operations are uniquely determined by the equations

$$x \backslash y = \bigvee \{z \in \mathcal{Q} : x \cdot z \leqslant y\} \tag{3}$$

and

$$y/x = \bigvee \{z \in \mathcal{Q} : z \cdot x \leqslant y\}. \tag{4}$$

If the quantale $\mathcal{Q}$ is commutative then the left and right residuals coincide, i.e., the equation $x \backslash y = y/x$ holds true for all $x, y \in \mathcal{Q}$.

Example 1. The set of extended nonnegative real numbers $\mathbb{R}^{\geqslant 0}_{+\infty} = [0, +\infty]$ represents a complete lattice with the usual ordering. The largest and the least elements are $+\infty$ and 0, respectively.