Signal Processing: *Image Communication* 35 (2015) 61–70

Contents lists available at ScienceDirect

Signal Processing: *Image Communication*

journal homepage: www.elsevier.com/locate/image

Undecimated Dual-Tree Complex Wavelet Transforms

P.R. Hill [a,*], N. Anantrasirichai [a], A. Achim [a], M.E. Al-Mualla [b], D.R. Bull [a]

[a] *Department of Electrical and Electronic Engineering, The University of Bristol, Bristol BS8 1UB, UK*
[b] *Khalifa University of Science, Technology and Research (KUSTAR) P.O. Box 127788, Abu Dhabi, UAE*

A R T I C L E I N F O

Article history:
Received 1 December 2014
Received in revised form
28 April 2015
Accepted 29 April 2015
Available online 9 May 2015

Keywords:
Discrete Wavelet Transforms
Image denoising

A B S T R A C T

Two undecimated forms of the Dual Tree Complex Wavelet Transform (DT-CWT) are introduced together with their application to image denoising and robust feature extraction. These undecimated transforms extend the DT-CWT through the removal of downsampling of filter outputs together with upsampling of the complex filter pairs in a similar structure to the Undecimated Discrete Wavelet Transform (UDWT).

Both developed transforms offer exact translational invariance, improved scale-to-scale coefficient correlation together with the directional selectivity of the DT-CWT. Additionally, within each developed transform, the subbands are of a consistent size. They therefore benefit from a direct one-to-one relationship between co-located coefficients at all scales and therefore this offers consistent phase relationships across scales. These advantages can be exploited within applications such as denoising, image fusion, segmentation and robust feature extraction. The results of two example applications (bivariate shrinkage denoising and robust feature extraction) demonstrate objective and subjective improvements over the DT-CWT. The two novel transforms together with the DT-CWT offer a trade-off between denoising performance, computational efficiency and memory requirements.

1. Introduction

The Discrete Wavelet Transform (DWT) has been used extensively for analysis, denoising and fusion within image processing applications [1–12]. It has been recognised that the DWT suffers from shift variance [13–15]. Variations on the DWT have been developed (e.g. cycle spinning introduced by [16]) to produce a shift invariant form. Exact shift invariance has been achieved using the Undecimated Discrete Wavelet Transform (UDWT). However, the UDWT variant suffers from a considerably over-complete representation together with a lack of directional selectivity. More recently, the Dual Tree Complex Wavelet Transform (DT-CWT) has offered a more compact representation while providing near shift invariance. The DT-CWT also offers improved directional selectivity (6 directional subbands per scale) and complex valued coefficients that are useful for magnitude/phase analysis within the transform domain.

Although overcomplete by a factor of two, the DT-CWT is still a compact transform (due to the downsampling at each level of the transform). However, non-compressive applications (such as denoising and fusion) do not require compact transforms and may benefit from overcomplete representations. Furthermore, not only does the Primary Visual Cortex (V1) contain filters that resemble 2D DT-CWT basis functions, it has been noted that the output from V1 is significantly overcomplete [17] (approximately 25:1 in area 17 [18]).

This paper proposes two undecimated forms of the DT-CWT which combine the benefits of the UDWT (exact translational invariance, a one-to-one relationship between all co-located coefficients at all scales) and the DT-CWT (improved directional selectivity and complex subbands).

* Corresponding author. Tel.: +44 1179545192.
E-mail address: paul.hill@bristol.ac.uk (P.R. Hill).

http://dx.doi.org/10.1016/j.image.2015.04.010

This paper builds on the previous work by Hill et al. [19] adding a second undecimated transform (U2DTCWT), a detailed analysis of both transforms, and a comprehensive list of additional results.

Firstly, the undecimated filter structure of the UDWT is introduced in Section 2. This filter structure is then applied to the DT-CWT decomposition to give the two forms of the undecimated DT-CWT as also described within Section 2. The shift invariance, generalisation to two dimensions and cross scale correlation of the developed transforms are then investigated in Sections 3.1, 3.2 and 3.3 respectively. Two applications of the transforms are described in Sections 4 and 5. Finally, a comparison of transforms and conclusion are presented in Sections 6 and 7 respectively.

2. Undecimated Wavelet Transforms

The undecimated form of the Discrete Wavelet Transform has been independently developed by numerous researchers separately referred to as "algorithme a trous" [14], the "Shift Invariant DWT (SIDWT)" [13], the "Stationary Wavelet Transform (SWT)" [20] and "Discrete Wavelet Frames (DWFs)" [15]. An overview of these separate developments is given by Fowler [21].

Defining the scaling and wavelet filters of an orthonormal DWT as $h \in \ell^2(\mathbb{Z})$ and $g \in \ell^2(\mathbb{Z})$ respectively, the undecimated wavelet filter (g) at scale $l+1$ is defined recursively as

$$g^{(l+1)}[k] = g^{(l)}[k]\uparrow 2 = \begin{cases} g^{(l)}\left[\frac{k}{2}\right] & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases} \tag{1}$$

where the filter $h^{(l+1)}$ is similarly defined. The downsampling at each stage of the DWT is removed to give the UDWT. The shift variation of the DWT, referred to earlier, is caused by this subsampling; its removal within the UDWT provides perfect shift invariance. It should be noted that each subband is now the same size as the original signal, leading to a considerably over-complete representation.

2.1. *The Undecimated Dual Tree Complex Wavelet Transform type 1: U1DT-CWT*

Although the UDWT is exactly shift invariant, it lacks directionality and has only real coefficients for analysis and processing. Kingsbury [2,22,23] formulated the Dual Tree Complex Wavelet Transform (DT-CWT) to provide near shift invariance and improved directionality with a more compact representation. The structure of the DT-CWT uses two separate trees to form Hilbert filter pairs within each subband. The magnitude response of this pair of filters is very close to being shift invariant. Another benefit of this transform is improved directional resolution (in the two dimensional version). These properties have underpinned excellent results in denoising, fusion and other image processing applications (e.g. [3,5,7]).

In common with conventional wavelet transforms, the size of the subbands in the DT-CWT decreases in octave steps as the scale of the transform increases. This provides a trade-off between resolution and redundancy at each level. However, this does not lead to a one-to-one relationship between co-located coefficients across scales. Cross scale relationships are exploited in segmentation, fusion and denoising applications (e.g. [24,25]). The correlation between a coefficient and its parent is utilised within each of these applications.

Although subsampling is justified for compression applications, the subsampled subbands of the DT-CWT have a restricted number of coefficients directly related to each spatial position in the signal or image, a relationship that often conflicts with the requirements of analysis applications [25]. To enable such analysis we now define an undecimated form of the DT-CWT, *the U1DT-CWT*, where each subband has the same resolution as the signal. As the U1DT-CWT contains no subsampling it exhibits perfect shift invariance. It also offers a one-to-one relationship between all co-located coefficients and the original samples and between co-located coefficients across all subbands.

The U1DT-CWT analysis stage is shown in Fig. 1. Filters at each stage are based on the filters used in the DT-CWT.

- Any perfect reconstruction bi-orthogonal set of filters can be used for the first level. The filters of one tree ($g0_0, g0_1$) are exactly the same as the other tree ($h0_0, h0_1$) but offset by one sample. These filters are not upsampled using the method illustrated by (2) (they do not need to take into account a previous level's subsampling).
- All subsequent filters in both trees are based on the *Q*-shift filters defined in [2].

The subsampling of the DT-CWT has been removed as indicated by the crosses in Fig. 1. To offset this effect at the second and subsequent levels, each of the *Q*-shift filters is itself upsampled (i.e. zeros are inserted between filter coefficients).

For example, the *Q*-shift filter $g1_0$ in the U1DT-CWT at stage $l+1$ is defined recursively (similarly to (1) for the UDWT) as

$$g1_0^{(l+1)}[k] = g1_0^{(l)}[k]\uparrow 2 = \begin{cases} g1_0^{(l)}\left[\frac{k}{2}\right] & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases} \tag{2}$$

where $g1_0^{(1)}$ is the original (non-upsampled) *Q*-shift filter. The other upsampled filters (at stage $l+1$: $g1_1^{(l+1)}$, $h1_0^{(l+1)}$ and $h1_1^{(l+1)}$) in Fig. 1 are similarly defined (and for all subsequent stages). The synthesis stage of the whole

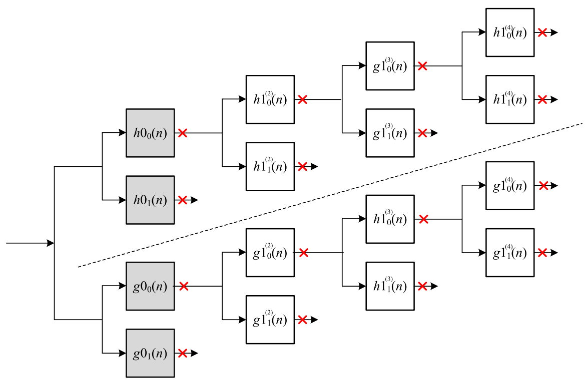

Fig. 1. The analysis stage of the U1DT-CWT. The crosses indicate the positions where downsampling would normally occur within the decimated DT-CWT.