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Homographic p-norms: Metrics of homographic image transformation

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ABSTRACT

Images often need to be aligned in a single coordinate system, and homography is one of the most efficient geometric models to align images. This paper presents homographic p-norms, scalar metrics of homographic image transformation, and to the best of our knowledge these are the most rigorous definition of scalar metrics quantifying homographic image transformations. We first define a metric between two homography matrices, and show it satisfies metric properties. Then we propose metrics of a single homography matrix for a general planar region, and ones for a usual rectangle image. For use of the proposed metrics, we provide useful homographic 2-norm expressions derived from the definition of the metrics, and compare the approximation errors of the metrics with respect to the exact metric. As a result, the discrete version of the metric obtained by pixel-wise computation is greatly close to the exact metric. The proposed metrics can be applied to evaluation of transformation magnitude, image closeness estimation, evaluation of camera pose difference, selection of image pair in stereo vision, panoramic image mosaic, and deblurring. Experimental results show the efficacy of the proposed metrics.

1. Introduction

Images often need to be aligned [1] in a single coordinate system to acquire 3D information, to detect/measure geometric difference, or to increase the FoV (field of view) or SNR (signal to noise ratio). Homography [2] is one of the popular geometric transformation models, and is frequently used in alignment of images and related fields [3–5]. Homography is a combination of several simpler transformations such as translation, rotation, (an)isotropic scaling, and perspectivity. It is generally expressed as a 3×3 matrix, but is effectively represented by eight parameters. Homographic transformation of an image can be implemented by multiplying the image coordinates with the homography transform matrix and normalizing the first and second components by the third component of the resulting vector.

Scalar metrics of non-scalar tensors such as vectors and matrices are useful for simplified representation of the non-scalar tensors and related tasks (e.g., force computation from penetration depth [6]), and they are usually called norms, e.g., vector norms and matrix norms. To the best of our knowledge, there have never been any scalar metric representing the amount of the homographic image transformation. The existing p-norm $\| \cdot \|_p \equiv \left(\sum | \cdot |^p\right)^{1/p}$, deals with every components or entries equally, and that is not proper for homography matrix, whose entries generally have different amounts of contribution in image transformations from each other.

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We define the magnitude of homographic transformation of an image as the p-norm of the displacements occurred by the transformation of the image, i.e., the p-norm of the displacements of the image coordinates. We first define homographic p-norm distance between two homography matrices, and then homographic p-norm of a single homography matrix for a planar region. Metric properties are proven for a general planar region, and metrics for a usual rectangle image and scale-invariant metrics are proposed. If the magnitude of homographic image transformation is defined for discontinuous coordinates, the exact magnitude can be computed only by considering the transformation of every pixel coordinates of the image. Hence we define the magnitude of the homographic transformation analytically for continuous coordinates.

For a usual $m \times n$ image, where m and n denote the width and the height, respectively, we consider 2D coordinates in the rectangular interval $(0,0) \le (x,y) \le (m,n)$. Integrating p-norm of the homographic displacement with respect to the image coordinates, the metric might be explicitly computed. In particular, we provide several explicit expressions for homographic 2-norms. The resulting expressions may be highly complex. However for an image of usual or high resolution, its computation is extremely faster than the estimation by summing up all 2-norms of displacements of the image.

Instead of using formal homographic p-norm metrics, one may compute the average distance for several pairs of feature locations in two images, obtained by an appropriate descriptor such as Harris detector [7], SIFT [8], SURF [9], and HOG [10]. However, the average distance for those features is far different from the average distance for the whole images, since features are not fairly distributed in images. Zhang presented a method to measure the difference between two fundamental matrices [11]. The method computes the average distance from randomly selected points on the epipolar lines defined by one fundamental matrix and randomly selected points in an image, to the epipolar lines defined by the other fundamental matrix and the randomly selected points in the image. Since the method depends on point selection, the resulting average distance is much different from exact metrics.

The metrics of homographic image transformation can be applied to evaluation of transformation magnitude, image closeness estimation, evaluation of camera pose difference, selection of image pair in stereo vision [12–14], panoramic image mosaic [15,3,16], and deblurring [17,18].

The rest of the paper is organized as follows. In Section 2 we define homographic p-norms for a general planar region. In Section 3 we present homographic p-norms for a usual rectangle image, and Section 4 provides explicit expressions of homographic 2-norms. Section 5 compares approximation errors of the proposed metrics. Section 6 gives discussion and implication. Experimental results are demonstrated in Section 7. Section 8 concludes the paper.

2. Homographic p-norms for planar region

This section defines homographic p-norm metrics for a general planar region. For a position $\mathbf{p} = (x,y)^{\mathrm{T}} \in \mathbb{R}^2$, let $\mathbf{p}^{\mathbf{H}}$ be its homographic transform by a homography matrix $\mathbf{H} = \eta \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix}$, where $\eta \in \mathbb{R}$ is assumed to be nonzero. For the sake of simplicity, this paper considers only finite coordinates, hence we assume $gx+hy+1 \neq 0$. Then $\mathbf{p}^{\mathbf{H}} = \left(\frac{ax+by+c}{gx+hy+1}, \frac{dx+ey+f}{gx+hy+1}\right)^{\mathrm{T}}$, and $\mathbf{p}^{\mathbf{H}}$ is invariant to η. Table 1 summarizes the notation used in the paper.

2.1. Homographic p-norm distance between two homography matrices

Let $S \subset \mathbb{R}^2$ be a planar region whose area is nonzero. We define a homographic p-norm distance between two homography matrices $\mathbf{H}^{\mathfrak{A}}$ and $\mathbf{H}^{\mathfrak{B}}$ for S as the p-norm of the inter-displacements between the two homographic transforms of the region S by $\mathbf{H}^{\mathfrak{A}}$ and $\mathbf{H}^{\mathfrak{B}}$

$$L_p^S\left(\mathbf{H}^{\mathfrak{A}}, \mathbf{H}^{\mathfrak{B}}\right) \equiv \left(\int_S \| \mathbf{p}^{\mathbf{H}^{\mathfrak{A}}} - \mathbf{p}^{\mathbf{H}^{\mathfrak{B}}} \|^p \mathrm{d}S\right)^{1/p}, \tag{1}$$

where $1 \le p < \infty$, and $\| \cdot \|^p \equiv \left(\| \cdot \|_p\right)^p = \sum | \cdot |^p$ (i.e., p-norm to the p). Then, the following lemma holds.

Lemma 1. *For homography matrices* $\mathbf{H}^{\mathfrak{A}}$, $\mathbf{H}^{\mathfrak{B}}$, *and* $\mathbf{H}^{\mathfrak{C}}$, *the above metric has the following properties*:

1. *Non-negativity*. $L_p^S\left(\mathbf{H}^{\mathfrak{A}}, \mathbf{H}^{\mathfrak{B}}\right) \ge 0$.
2. *Nondegeneracy*. $L_p^S\left(\mathbf{H}^{\mathfrak{A}}, \mathbf{H}^{\mathfrak{B}}\right) = 0$ *if and only if* $\mathbf{H}^{\mathfrak{A}} = \zeta \mathbf{H}^{\mathfrak{B}}$, *where* $\zeta \in \mathbb{R}$ *is nonzero.*
3. *Symmetry*. $L_p^S\left(\mathbf{H}^{\mathfrak{A}}, \mathbf{H}^{\mathfrak{B}}\right) = L_p^S\left(\mathbf{H}^{\mathfrak{B}}, \mathbf{H}^{\mathfrak{A}}\right)$.
4. *Triangle inequality*. $L_p^S\left(\mathbf{H}^{\mathfrak{A}}, \mathbf{H}^{\mathfrak{C}}\right) \le L_p^S\left(\mathbf{H}^{\mathfrak{A}}, \mathbf{H}^{\mathfrak{B}}\right) + L_p^S\left(\mathbf{H}^{\mathfrak{B}}, \mathbf{H}^{\mathfrak{C}}\right)$.