Stability of periodic solutions of periodic cellular neural networks with delays and impulsive perturbations

Kaihua Wang, Yushu Sun
School of Mathematics and Statistics
Hainan Normal University
Haikou, Hainan, 571158, P.R. China
kaihuawang@qq.com, 35225454@qq.com

Zhanji Gui (Corresponding author)
Department of Software Engineering
Hainan College of Software Technology
Qionghai, Hainan, 571400, P.R. China
zhanjigui@sohu.com

***Abstract*—in this paper, we consider the dynamics of an artificial periodic cellular neural networks with time delays and impulsive control. Sufficient conditions are obtained for the stability of a unique periodic solution. The results extend earlier ones where impulse is absent. Further, the influences of the impulsive effect are investigated by using computer simulation method. These results are helpful to design globally asymptotically stable and periodic oscillatory cellular neural networks with time delay and impulsive perturbations.**

***Keywords*—global asymptotical stability; Periodic solution; Delay; Impulsive; Cellular neural networks.**

I. INTRODUCTION

Cellular neural networks, which are widely applied in many fields such as signal processing, pattern recognition, optimization, etc. are attached more and more interesting by many researchers ([1-4]). In recent years, considerable attention has been paid to study the dynamics of neural networks. Many essential features of these networks, such as qualitative properties of the existence of equilibrium point or periodic solution, global asymptotic stability and global exponential stability etc. have been widely investigated (see [5-9]).

On the other hand, in real world, many evolutionary processes are characterized by abrupt changes at a certain time or with time delays. These changes are called to be impulsive phenomena and time delay phenomena respectively, which are included in many fields such as physics, chemistry, population dynamics, optimal control, etc. and can be described by impulsive differential equations and time delay differential equations. Thus, researches of neural networks with impulse or time lag have been received much interesting ([10–14]).

In this paper, we consider the following artificial periodic cellular neural networks model with variable time delay and impulsive perturbations:

$$\begin{cases} \dfrac{dx_i}{dt} = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t)) \\ \qquad + \sum_{j=1}^{n} b_{ij}(t) f_j(x_j(t-\tau_{ij}(t))) + I_i(t), \quad t \neq t_k, \\ \Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = J_k^i x_i(t_k), \quad t = t_k,\ k = 1,2,\ldots \end{cases} \tag{1}$$

where $i = 1,2,\ldots,n$. n corresponds to the number of units in a neural network; $x_i(t)$ corresponds to the state of the i th unit at time t; $f_j(x_j(t))$ denotes the output of the j th unit at time t; $a_{ij}(t)$ denotes the strength of the j th unit on the i th unit at time t; $b_{ij}(t)$ denotes the strength of the j th unit on the i th unit at time $t-\tau_{ij}(t)$; $I_i(t)$ is the external bias on the i th at time t; $\tau_{ij}(t)$ corresponds to the transmission delay along the axon of the j th unit; $c_i(t)$ represents the rate with which the i th unit will reset its potential to the resting state when disconnected from the network and external inputs; $\Delta x_i(t_k)$ are the impulses at moments t_k and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = +\infty$; J_k^i be impulsive inputs from outside the network and there exists a positive integer q such that $t_{k+q} = t_k + \omega$, $J_{k+q}^i = J_k^i$.

In this paper, we assume that the activation functions $f_i(x)$, $i = 1,2,\ldots,n$ satisfies the following conditions:

(H1) For each $i \in \{1,2,\ldots,n\}$, $f_i : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz with Lipschitz constant L_i, i.e. $|f_i(x_i) - f_i(y_i)| \le L_i |x_i - y_i|$ for all $x_i, y_i \in \mathbb{R}$.

(H2) For each $i \in \{1,2,\ldots,n\}$, $|f_i(x)| \le M_i$, $x \in \mathbb{R}$, for some constant $M_i > 0$.

Throughout this paper, we always assume that $a_{ij}(t)$, $b_{ij}(t)$ and $I_i(t)$ are continuous ω-periodic functions. $\tau_{ij}(t) \ge 0$ are continuously differentiable ω-periodic functions with $0 \le \tau'_{ij}(t) < 1$, $c_i(t)$ are positive continuous ω-periodic functions. If $g(t)$ is an ω-periodic functions, for the sake of convenience, we introduce the following notations:

$$g^+ = \max_{0\le t\le\omega} |g(t)|,\ g^- = \min_{0\le t\le\omega} |g(t)|$$

II. GLOBAL ASYMPTOTIC STABILITY OF PERIODIC SOLUTION

By using the similar process as in [15], we can use Mawhin's continuity theorem [16] to prove the following theorem about the existence of periodic solution of system .

Theorem l Assume that (H1)-(H2) holds and

(H₃) D is a nonsingular M-matrix,

where $D=(d_{ij})_{n\times n}$,

$$d_{ij}=\delta_{ij}-\frac{b_{ij}^{+}L_j\omega}{\sqrt{2}}\left(1+\frac{c_i^{+}}{c_i^{-}}\right),\quad \delta_{ij}=\begin{cases}1, & i=j,\\ 0, & i\neq j.\end{cases}$$

Then system has at least one ω-periodic solution.

Next, we shall constructing some suitable Liapunov functions to derive the sufficient conditions which ensure that has a unique global asymptotic stability ω-periodic solution.

Let $x^*(t)$ be an ω-periodic solution of our cellular neural networks model and $x(t)$ be any solution of . Set $u(t)=x(t)-x^*(t)$. Substituting $x(t)=u(t)+x^*(t)$ into leads to

$$\begin{cases}\dfrac{du_i}{dt}=-c_i(t)u_i(t)+\displaystyle\sum_{j=1}^{n}a_{ij}(t)F_j(u_j(t))\\ \qquad +\displaystyle\sum_{j=1}^{n}b_{ij}(t)F_j(u_j(t-\tau_{ij}(t))),\quad t\neq t_k,\\ u_i(t_k^{+})=(1+J_k^{i})u_i(t_k),\quad t=t_k,\ k=1,2,\ldots,\end{cases}\tag{2}$$

where $F_j(u_j(t))=f_j(u_j(t)+x_j^*(t))-f_j(x_j^*(t))$. By (H₁), we have

$$|F_j(u_j(t))|\leq L_j\,|u_j(t)|,$$
$$|F_j(u_j(t-\tau_{ij}(t)))|\leq L_j\,|u_j(t-\tau_{ij}(t))|.$$

Set $p_{ij}^{m}=\max\left\{\frac{1}{1-\tau'_{ij}(t)}:t\in\mathbb{R}\right\}$, for $i,j=1,2,\ldots,n$. Due to $\tau'_{ij}(t)<1$, $t\in\mathbb{R}$ and periodicity of these functions, it is obvious that p_{ij}^{m} are positive constants. We define

$$\sigma_{ij}(t)=t-\tau'_{ij}(t),\quad t\in\mathbb{R},$$

for $i,j=1,2,\ldots,n$. It follows from $\tau'_{ij}(t)<1$ for all $t\in\mathbb{R}$, that $\sigma_{ij}(t)$ has its inverse function ν_{ij}. Now we are in position to present our main result in this paper.

Theorem 2 Assume that (H₁)-(H₃) hold. Furthermore, assume that

(H₄) $\Gamma_i:=2c_i^{-}-\sum_{j=1}^{n}\left(a_{ij}^{+}+b_{ij}^{+}\right)L_j-L_i\sum_{j=1}^{n}\left(a_{ji}^{+}+b_{ji}^{+}p_{ji}^{m}\right)>0$,

then system has a unique ω-periodic solution which is globally asymptotically stable.

Proof. We consider the Liapunov function

$$V(t)=V_1(t)+V_2(t)\tag{3}$$

where

$$V_1(t)=\sum_{i=1}^{n}u_i^2(t),\quad V_2(t)=\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}^{+}L_j\int_{t-\tau_{ij}(t)}^{t}\frac{u_j^2(s)}{1-\tau'_{ij}(\nu_{ij}(s))}ds$$

Calculating the upper right derivatives of the V_1 and V_2 along the solution of respectively, we get

$$\begin{aligned}D^{+}V_1(t)\Big|_{(2)}&=2\sum_{i=1}^{n}u_i(t)\Bigg[-c_i(t)u_i(t)+\sum_{j=1}^{n}a_{ij}(t)F_j(u_j(t))\\&\qquad+\sum_{j=1}^{n}b_{ij}(t)F_j(u_j(t-\tau_{ij}(t)))\Bigg]\\&\leq\sum_{i=1}^{n}\Bigg[-2c_i^{-}u_i^2(t)+\sum_{j=1}^{n}2a_{ij}^{+}\,|u_i(t)|\,|F_j(u_j(t))|\\&\qquad+\sum_{j=1}^{n}2b_{ij}^{+}\,|u_i(t)|\,|F_j(u_j(t-\tau_{ij}(t)))|\Bigg]\\&\leq\sum_{i=1}^{n}\Bigg[-2c_i^{-}u_i^2(t)+\sum_{j=1}^{n}2a_{ij}^{+}L_j\,|u_i(t)|\,|u_j(t)|\\&\qquad+\sum_{j=1}^{n}2b_{ij}^{+}L_j\,|u_i(t)|\,|u_j(t-\tau_{ij}(t))|\Bigg]\\&\leq\sum_{i=1}^{n}\Bigg[-2c_i^{-}u_i^2(t)+\sum_{j=1}^{n}a_{ij}^{+}L_ju_i^2(t)+\sum_{j=1}^{n}a_{ij}^{+}L_ju_j^2(t)\\&\qquad+\sum_{j=1}^{n}b_{ij}^{+}L_ju_i^2(t)+\sum_{j=1}^{n}b_{ij}^{+}L_ju_j^2(t-\tau_{ij}(t))\Bigg]\end{aligned}\tag{4}$$

and

$$D^{+}V_2(t)\Big|_{(2)}=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{b_{ij}^{+}L_j}{1-\tau'_{ij}(\nu_{ij}(t))}u_j^2(t)-\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}^{+}L_ju_j^2(t-\tau_{ij}(t))\tag{5}$$

It follows from - and (H₄) that

$$\begin{aligned}D^{+}V(t)\Big|&\leq\sum_{i=1}^{n}\Bigg[-2c_i^{-}u_i^2(t)+\sum_{j=1}^{n}\left(a_{ij}^{+}L_j+b_{ij}^{+}L_j\right)u_i^2(t)\\&\qquad+\sum_{j=1}^{n}\left(a_{ij}^{+}L_j+b_{ij}^{+}p_{ij}^{m}L_j\right)u_j^2(t)\Bigg]\\&\leq\sum_{i=1}^{n}\Bigg[-2c_i^{-}+\sum_{j=1}^{n}\left(a_{ij}^{+}+b_{ij}^{+}\right)L_j\\&\leq+L_i\sum_{j=1}^{n}\left(a_{ji}^{+}+b_{ji}^{+}p_{ji}^{m}\right)\Bigg]u_i^2(t)\leq 0\end{aligned}\tag{6}$$

On the other hand, for $t=t_k$, we have

$$\begin{aligned}V(t_k^{+})&=\sum_{i=1}^{n}((1+J_k^{i})u_i(t_k))^2\\&\qquad+\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}^{+}L_j\int_{t_k^{+}-\tau_{ij}(t_k^{+})}^{t_k^{+}}\frac{u_j^2(s)}{1-\tau'_{ij}(\nu_{ij}(s))}ds\\&\leq\sum_{i=1}^{n}u_i^2(t_k)+\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}^{+}L_j\int_{t-\tau_{ij}(t_k)}^{t}\frac{u_j^2(s)}{1-\tau'_{ij}(\nu_{ij}(s))}ds\\&=V(t_k)\end{aligned}\tag{7}$$

From and , we have