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Convex optimisation of gradient and shim coil winding patterns

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ABSTRACT

Gradient and shim coils were designed using boundary element methods with convex optimisation. The convex optimisation framework permits the prototyping of many different cost functions and constraints, for example ℓ^p-norms of the current density. Several examples of gradients and shims were designed and simulated to demonstrate this, as well as to investigate the behaviour of new cost functions. A mixture of ℓ^1- and ℓ^∞-norms of the current density, when used as a regularisation term in the field synthesis problem, was found to produce coils with bunches of equally spaced windings that do not take up all of the available surface. This is thought to be beneficial in the design of coils that will be manufactured from wire with a fixed cross-section.

1. Introduction

Magnetic resonance imaging requires 3 different types of magnetic field produced by the main magnet, gradient and radio-frequency (RF) coils. The main magnet generates an exquisitely homogeneous, stable and intense magnetic field usually with superconducting wires and is designed with a great deal of engineering knowledge and experience [1]. Radio-frequency coils are usually simple rectangular or circular loops combined with capacitors in circuits resonant at the Larmor frequency. Gradient coils must produce their magnetic fields to within a few % in the imaging region and are made from coils of copper wire at room temperature. Gradient coils must operate safely in the audio frequency range up to ~20 kHz and support a maximum current of more than 800 A with more than 2000 V potential difference across each terminal. Shim coils are similar to gradient coils but need not support such high current and voltages. They are used to correct the magnetic field of the main magnet.

The design of gradient and shim coils [2] can be divided into two types of methods, discrete wire and current density method, each with arguments for and against their use. Discrete wire methods model the thin wires that constitute a coil and continuous current density methods model a coil with a surface current density.

In discrete methods, the line-integral form of the Biot–Savart law is used to calculate the magnetic field produced by a coil. The position of the wires is parameterised and a cost function that defines the "quality" of the coil [3,4] is usually non-linear. Therefore, it is common and entirely appropriate to use stochastic optimisation techniques for global optimisation of the cost function with discrete wire coil parameterisation [5]. It is conceptually simple to add various kinds of penalty to the cost function to yield the most desirable design. However, a new parameterisation is required when the coil surface is changed requiring considerable mathematical effort for complicated surface shapes [6].

Continuous current density coil design methods [7,8] provide a good approximation to the manufactured coil if enough wires are used. A thin surface with a current density flowing on it can be modelled as incompressible flow with zero divergence. The surface integral form of the Biot–Savart law is used to calculate the magnetic field produced by a coil. Various parameterisations of continuous surface current density are possible that provide a simple linear relationship with the magnetic induction field. The most versatile of these is a parameterisation in which the current carrying surface is modelled as a connected ensemble of flat polygons, known as a mesh. The current density in the surface is then a vector field that is piecewise uniform [9]. The number of free

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parameters over which to optimise can be greatly reduced with assumptions of symmetry [10,11]. Inversion of the Biot–Savart law to obtain coil designs is ill-posed, but easily solved by weighted Tikhonov regularisation with the power dissipation [7,12] or stored energy [13] of the coil with one-step matrix inversion methods.

Since the current density has zero divergence and is confined to a surface, a potential function, the stream-function, is defined over the surface [8]. A scalar function whose curl (about the surface normal) yields the current density, the stream function is piecewise linear over a polygonal mesh [9,14]. The vertices of the mesh possess a stream function value that is linearly interpolated over its immediate neighbourhood of triangles. It is these stream function values that act as free parameters of continuous current density coil design methods. A final step in continuous current density methods is the conversion into discrete wires by taking level sets of the stream-function over the surface.

Cost functions involving non-quadratic functions of the free-parameters of the coil design have been studied previously and solved efficiently using custom gradient descent based deterministic optimisation algorithms [4,15–18]. In the present paper we introduce a convenient general formulation of the problem where all components of the optimisation are convex. Several prototyping tools are available for the solution of convex optimisation problems. In the present study we used cvx [19,20] and demonstrate how new cost functions are easily prototyped. In particular, the magnitude of the current density is a convex function of the nodal stream function values and therefore we can minimise a convex norm of the absolute current density. Previously the infinity norm of the gradient magnitude was included in the optimisation to provide coils with minimised maximum current density [15]. These coils exhibit a lower maximum temperature for a given field gradient strength [21]. It should be noted that while it has always been possible to add such terms to the cost function, it is only when formulated in this way that they can be solved efficiently and deterministically and associated in any combination.

This paper is arranged as follows: the mathematical formulation of the convex optimisation problem is given, followed by its representation in the parsed cvx code as written in Matlab (The Mathworks, Natick, MA, USA). The ℓ^1-, weighted ℓ^2- and ℓ^∞-norms of the current density were used for regularisation using triangular and sinusoidal boundary element methods (BEMs) on arbitrary and cylindrical surfaces, respectively. Fundamental studies (similar to those presented in Refs. [22,21]) of the behaviour of this new optimisation were made by trading the field accuracy for different norms of the current density and also by mixing them in varying amounts with fixed field accuracy. Coil patterns are shown for 0th, 1st and 2nd order solid harmonic target fields with resistance and mixed norm minimised solutions and shielded whole-body X-gradient coils were designed for comparison. Simulations of the resultant novel coil designs were performed and are presented herein with discussions of their relevance to coil construction.

2. Methods

2.1. Problem Formulation

Two types of BEM were used in this study. One with flat triangular elements and linear shape functions to model the stream-function of the current density [9,14,23]. The values of the stream-function at the nodes of the mesh, ψ, define the coil. Meshes were made in Blender (Blender Foundation, Amsterdam, Netherlands) and exported to Matlab. The second method is restricted to a finite-length cylindrical surface on which the stream-function is a weighted sum of truncated sinusoidal functions [10,11].

Matrix equations that transform ψ to the various coil properties were constructed according to Eqs. (9)–(13) in Ref. [15] and are summarised below.

- The z-component of the magnetic field at a series of points,
$$b = B\psi; \tag{1}$$
- The Cartesian or cylindrical components of the current density in each triangle,
$$j_x = J_x\psi, \quad j_y = J_y\psi, \quad j_z = J_z\psi, \quad j_{a\phi} = J_{a\phi}\psi; \tag{2}$$
- The stored energy in the coil;
$$W = \psi^T L_c \psi; \tag{3}$$
- The resistive power dissipation of the coil,
$$P = \psi^T R_c \psi. \tag{4}$$

Previously, a weighted linear combination optimisation problem was used with equality constraints. This is simple to solve when each term is quadratic in ψ using matrix inversion (with Lagrange multipliers for the equality constraints) [14,23]. A non-quadratic term was introduced to minimise the maximum absolute current density [15]. We generalise this by stating the optimisation in the form of a convex optimisation as defined in Ref. [24]:

$$\begin{aligned} &\text{minimise} \quad f_0(\psi) \\ &\text{subject to} \quad f_i(\psi) \leqslant b_i, \quad i = 1, \ldots, m, \end{aligned} \tag{5}$$

where the functions $f_0, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}$ are convex. Eqs. (1) and (2) are affine. Eqs. (3) and (4) are convex if L_c and R_c are positive semidefinite, which is the case.

Various convex cost functions and constraints were tested in the present study using cvx; a toolbox for Matlab that parses the optimisation problem and passes it in standard form to sdtp3 [25] to find the solution. The simplest optimisation for the coil design problem is the Tikhonov regularised minimisation of the root-mean-squared (RMS) residual field.

$$\text{minimise} \quad \|B\psi - b_t\|_2 + \alpha\|\psi\|_2, \tag{6}$$

where b_t is the target field and α is the user-defined regularisation parameter.

The ℓ^2-norm regularisation term, $\|\psi\|_2$, has no practical relevance. Substitution of ψ with an affine transformation $F\psi$ retains convexity. If F is the matrix obtained from Cholesky decomposition of L_c or R_c then the solution of Eq. (6) will yield, respectively, the inductance (stored energy) or resistance (power dissipation) minimised coil.

The minimum resistance coil design problem is written in cvx code as.

```
cvx_begin
   variable x(length(Rc));
   minimize (norm(B*x-bt,2) + alph*quad_form
  (x,Rc));
cvx_end
```

It should be noted that it is far simpler and quicker to solve the specific problem described by the code above using direct matrix inversion [9,14,23] than this convex optimisation method. In all cases, optimisations were performed on a Mac Pro (Apple Inc., Cupertino, CA, USA) equipped with 2.8 GHz Intel Xeon processors and 4 GB of RAM, using Matlab R2012a (The Mathworks, Natick, MA, USA), CVX [24] 2.0 (beta) build 945 and sdpt3 [25] version 4.0.