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# Coupled modes, frequencies and fields of a dielectric resonator and a cavity using coupled mode theory

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ABSTRACT

Probes consisting of a dielectric resonator (DR) inserted in a cavity are important integral components of electron paramagnetic resonance (EPR) spectrometers because of their high signal-to-noise ratio. This article studies the behavior of this system, based on the coupling between its dielectric and cavity modes. Coupled-mode theory (CMT) is used to determine the frequencies and electromagnetic fields of this coupled system. General expressions for the frequencies and field distributions are derived for both the resulting symmetric and anti-symmetric modes. These expressions are applicable to a wide range of frequencies (from MHz to THz). The coupling of cavities and DRs of various sizes and their resonant frequencies are studied in detail. Since the DR is situated within the cavity then the coupling between them is strong. In some cases the coupling coefficient, $\kappa$, is found to be as high as 0.4 even though the frequency difference between the uncoupled modes is large. This is directly attributed to the strong overlap between the fields of the uncoupled DR and cavity modes. In most cases, this improves the signal to noise ratio of the spectrometer. When the DR and the cavity have the same frequency, the coupled electromagnetic fields are found to contain equal contributions from the fields of the two uncoupled modes. This situation is ideal for the excitation of the probe through an iris on the cavity wall. To verify and validate the results, finite element simulations are carried out. This is achieved by simulating the coupling between a cylindrical cavity's $TE_{011}$ and the dielectric insert's $TE_{01\delta}$ modes. Coupling between the modes of higher order is also investigated and discussed. Based on CMT, closed form expressions for the fields of the coupled system are proposed. These expressions are crucial in the analysis of the probe's performance.



## 1. Introduction

The coupling between a dielectric resonator (DR) and a conducting cavity is of interest in electron paramagnetic resonance (EPR) spectroscopy because of the signal to noise ratio (SNR) enhancement of the resulting probe [1–4]. When both resonators have the same resonance frequency, the size of the DR is much smaller than that of the cavity. Therefore the magnetic field of the DR is more concentrated in a much smaller spatial region. This leads to an increase in the resonator's filling factor [4]. Usually the frequency of the dielectric $TE_{01\delta}$ mode and that of the rectangular cavity's $TE_{102}$ mode ($TE_{011}$ mode for cylindrical cavities) are close. Two DRs inserted in a cavity allow the user to tune the frequency of the cavity along with enhancing the SNR [3,5,6].

The coupling between the $TE_{01\delta}$ DR mode and the cavity's $TE_{011}$ mode was studied by Mett et al. [7]. They showed that the coupling could be modeled by lumped circuit (LC) elements. Using the LC model, crucial probe parameters such as frequencies, quality factors and resonator efficiencies were determined. The interaction of the dielectric and cavity modes results in two new modes. A symmetric (parallel) and an anti-symmetric (anti parallel) mode [7]. The symmetric mode is the mode formed when the two electromagnetic fields add constructively in phase, while the anti-symmetric mode has a 180° phase shift between the two uncoupled modes.

Using finite element simulations, the current authors showed that the interaction between the $TE_{01\delta}$ modes of two DRs and a $TE_{102}$ cavity mode results in three coupled modes, where the most appropriate mode for X-band EPR experiments was found to be the $TE^{+++}$ mode [6]. This mode is the result of the in-phase coupling of the three uncoupled ones (the two $TE_{01\delta}$ Dielectric modes and the $TE_{102}$ cavity mode). In fact, it was illustrated that the fields of the $TE^{+++}$ mode is the linear superposition of the three uncoupled ones [6]. The $TE^{+--}$ mode does not have a cavity contribution. Accordingly, this mode is very difficult to excite through the cavity iris. As noted, the behavior of the coupled modes varies significantly. Thus for EPR experiments one needs to have a comprehensive

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understanding of the modes' characteristics, particularly their frequencies and field distributions. Other field dependent parameters are of interest as well. Accordingly, the aim of the current paper is to thoroughly study the interaction of the modes of a single DR and an enclosing cavity. Coupled frequencies as well as field distributions are analytically determined.

Generally, coupled mode theory (CMT) is used to analyze and predict the behavior of a compound system by using the known properties of its simpler components. CMT can be divided into two main branches [8]. Space-coupled mode theory is useful in studying the properties of transmission systems such as waveguides and fiber optical systems. On the other hand, temporal-coupled mode theory is crucial in understanding the interaction between multiple resonators and is therefore suitable for the current work.

Usually CMT is applied to determine the frequencies (eigenvalues) of the coupled system. In the current manuscript, this step is taken further by calculating the coupled fields (eigenvectors) too. This is achieved by formulating the coupled mode equations from the first principles, i.e. from Maxwell's equations. By knowing the coupled frequencies and the fields of the coupled systems, a better understanding of probes with inserts can be achieved. Therefore, in this article, general expressions for the eigenvalues (frequencies) and the eigenvectors (fields) are calculated. The case when both resonators have the same uncoupled frequency (degenerate) is studied in detail. Other situations when the frequencies of the two subsystems are not the same are also thoroughly investigated. The results predicted by the coupled mode theory are compared to those of an electromagnetic (EM) full-wave numerical finite element simulator as well as to those found in the literature [7–9].

Section 2 defines the system and problem in a concise sense where the notations and the electric fields are presented. The eigenvalues and eigenvectors are derived in Section 3. This constitutes the main core needed to study the coupled system. Section 4 investigates and discusses the results obtained for different cases. The results are verified using finite element simulation. Summary and conclusions are presented in Section 5.

## 2. Theoretical background

The system under consideration is shown in Fig. 1. It consists of a DR, referred to as "1", inserted in the center of a cylindrical cavity, referred to as "2". The holder, not shown in the figure, is of a low loss/low permittivity material so its effect is negligible. Two types of DRs with vastly different dielectric constants were used. The first, labeled *type I, with* $\varepsilon_r = 29.2$, $d_1 = 6$ *mm*, $l_1 = 2.65$ *mm and* $f \approx 9.7$ GHz. The second labeled, *type II*, with $\varepsilon_r = 261$ $d_1 = 1.75$ mm, $l_1 = 1.75$ mm and $f \approx 9.5$ GHz. The terms $d$ and $l$ are the resonators diameter and height respectively. The cavity has an aspect ratio of $d_2/l_2 = 1$.

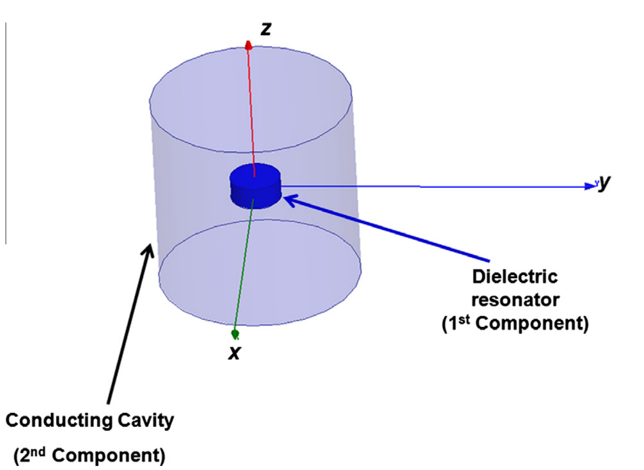


**Fig. 1.** System consisting of a DR inserted in a conducting cavity. The dielectric insert is held inside a hollow low loss/low permittivity holder (not shown).

The two uncoupled modes of interest are the dielectric $TE_{01\delta}$ mode and the cavity $TE_{011}$ mode. The electric field of each mode can be written as [10],

$$E_{\varphi 1} = M_1 J_1(k_1 r) \begin{cases} \cos(\beta z) & r \leqslant \frac{d_1}{2},\ |z| \leqslant \frac{l_1}{2} \\ e^{\frac{\alpha l_1}{2}} \cos\left(\frac{\beta l_1}{2}\right) e^{-\alpha |z|} & r < \frac{d_1}{2},\ |z| > \frac{l_1}{2} \end{cases} \tag{1}$$

and

$$E_{\varphi 2} = M_2 J_1(k_2 r) \cos\left(\frac{\pi z}{d_2}\right). \tag{2}$$

Here $k_1, k_2$ are the dielectric and cavity radial wave numbers respectively. The symbol $\beta$ is the wave number inside the dielectric in the $z$ direction, $M_{1,2}$ are the fields' amplitudes and $E_{\varphi i}$ is the azimuthal electric field component. In deriving Eq. (1) a perfectly magnetic waveguide was assumed [11]. In addition, $k_1 = p_{01}/r_1$, where $p_{01} = 2.405$ and is the root of the 0th order Bessel function, $J_0(x)$. The symbol $\alpha$ is the attenuation factor, in the $z$ direction, outside the DR resonator.

The magnetic field, which is the primary quantity that identifies the performance of the probe, can be determined from the electric field using Maxwell's equations, i.e.

$$\mathbf{H} = -\frac{1}{j\omega\mu_0} \nabla \times \mathbf{E},$$

where $\mu_0$ is the permeability of free space and $\omega$ is the resonant frequency.

## 3. Theory

### 3.1. Derivation of the frequencies and fields by CMT

Using CMT, the fields of the coupled system are expressed as linear superposition of the uncoupled ones,

$$\mathbf{E} = a_1 \mathbf{E}_1 + a_2 \mathbf{E}_2, \tag{3}$$

$$\mathbf{H} = b_1 \mathbf{H}_1 + b_2 \mathbf{H}_2, \tag{4}$$

The isolated mode of the DR satisfies Maxwell's curl equations,

$$\nabla \times \mathbf{E}_1 = -j\omega_1 \mu_0 \mathbf{H}_1, \tag{5}$$

$$\nabla \times \mathbf{H}_1 = j\omega_1 \varepsilon_1 \mathbf{E}_1,$$

where

$$\varepsilon_1 = \begin{cases} \varepsilon_r \varepsilon_0 & \text{inside the dielectric material} \\ \varepsilon_0 & \text{otherwise.} \end{cases}$$

It incorporates the spatial variation due to the DR. The $\omega_1$ symbol is the dielectric mode angular frequency. The variables $\mathbf{E}_1$ and $\mathbf{H}_1$ are the electric and magnetic fields respectively. Similarly, the cavity mode satisfies

$$\nabla \times \mathbf{E}_2 = -j\omega_2 \mu_0 \mathbf{H}_2 \tag{6}$$

and

$$\nabla \times \mathbf{H}_2 = j\omega_2 \varepsilon_2 \mathbf{E}_2,$$

where $\varepsilon_2 = \varepsilon_0$ is the permittivity of free space.