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Weakly nonlinear analysis on radially growing interface with the effect of viscous normal stress

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ABSTRACT

Two-phase Hele-Shaw problem is studied as a model for a two-dimensional radially growing interface. Instead of the Young–Laplace equation, which has been employed in almost all previous studies concerning Hele-Shaw problem, we apply the normal stress balance and derive the new mode coupling equation for perturbation of the interface. In the present research, weakly nonlinear analysis is carried out and time evolution of the interfaces is numerically calculated. These numerical results suggest that our model reflects more exactly the nonlinear features of a radially growing interface, rather than the previous ones based on the Young–Laplace equation.

1. Introduction and model

From the growth of a snow crystal to the flow of shale gas, there is a wide variety of examples related with the two-dimensional interface growth. Such phenomena have been intensively investigated as Hele-Shaw problem [1–8]. Hele-Shaw problem is concerned with a time evolution of an interface in a Hele-Shaw cell, which consists of two parallel plates with narrow gap filled with viscous fluids. Though the formulation of the problem is quite simple, it covers various areas of research, for instance, pattern formation of the unstable interface [9,10], physical properties of liquids [11], enhanced oil recovery in the engineering [3,12], and integrable systems in the theoretical physics [13–16]. It is well known that if a viscosity of the displacing fluid is smaller than that of the displaced one, then the interface between the fluids becomes unstable, so called Saffman–Taylor instability [1]. The pattern formed by the unstable interface is referred to as the viscous fingering. Here we focus on the viscous fingering which is observed in a radially growing interface in a Hele-Shaw cell, and investigate a mathematical model.

Clearly, the assumption that the fluids are incompressible and irrotational implies that Hele-Shaw problem is equivalent to solving the Laplace equations. As boundary conditions for the Laplace equations, almost all previous studies have adopted the kinematic boundary condition and the Young–Laplace equation [1,2,17,18]. However, the validity of the Young–Laplace equation has been discussed in [19–21]. Park and Homsy firstly took the wetting effect of displacing fluid into account and added a correction term to the Young–Laplace equation [19]. For the rectilinear cell, Schwartz investigated the qualitative changes of the dispersion relations due to the wetting effect, which support to take account of this effect [20]. In the case of the radial cell, Maxworthy performed experimental investigations which tend to agree with the theoretical ones by considering this effect [21]. In this context, in the recent theoretical study [22], weakly nonlinear analysis was carried out based on the mode coupling equation with the wetting effect. There, it was suggested that the model with the boundary condition including the wetting effect reflects the nonlinear features of the viscous fingering more accurately than the one without this effect.

On the other hand, another kind of corrected boundary condition was individually proposed by Kim et al. [23] and Gadêlha and Miranda [24], which applied the normal stress balance at the interface instead of the Young–Laplace equation. However, the work by Kim et al. [23] was limited to the linear analysis, and did not give us enough answer to the pattern formed by an unstable interface. While the one by Gadêlha and Miranda [24] did not contain all of the correction terms, and hence it seems to be insufficient. Therefore, as a next step, it is quite natural to develop the nonlinear analysis based on the normal stress balance properly. From these backgrounds, we investigate the following points: (i) to derive the new mode coupling equation which includes the complete form of the normal stress balance; (ii) to compare the behaviors of the interfaces whether the normal stress balance is considered or not (see [25] for more detailed information). In these means, our approach can be a complement of the previous studies about weakly nonlinear mode coupling equation.

Our model is as depicted in Fig. 1, where the thickness of the Hele-Shaw cell is b, and viscosities of fluids 1 and 2 are denoted by μ_1 and μ_2, respectively. Hereafter, subindex i represents the inner ($i = 1$) and the outer ($i = 2$) fluid, and moreover $\boldsymbol{v}_i = \boldsymbol{v}_i(r, \theta, t)$, $p_i = p_i(r, \theta, t)$ and ∇ are the velocity vector, the pressure of fluid i, and the differential operator in the polar coordinates (r, θ), respectively. Here we assume that flows of the fluids in the Hele-Shaw cell obeys Darcy's law $\boldsymbol{v}_i = -(b^2 / 12\mu_i)\nabla p_i$. From the Darcy's law and the

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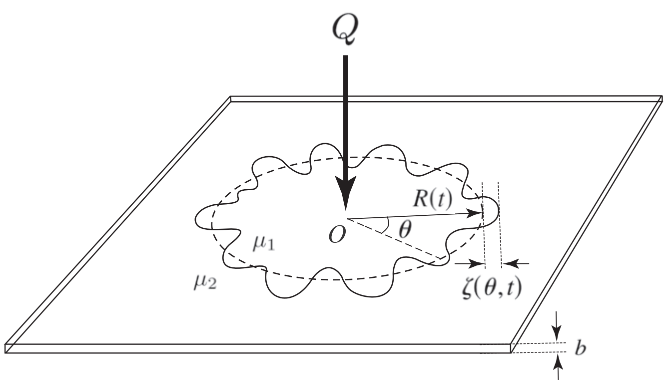

Fig. 1. Our model: HeleShaw cell and radially growing interface.

incompressibility of the fluids *i.e.*, $\nabla \cdot \boldsymbol{v}_i = 0$, it is derived that the velocity potential of each fluids satisfies the Laplace equations $\nabla^2\phi_i = 0$. Thus Hele-Shaw problem consists of solving the Laplace equations under the appropriate boundary conditions.

2. The derivation of the mode coupling equation

In this section we discuss the boundary conditions and derive the extended mode coupling equation. By following the work due to Miranda and coauthors [18,24], the general solution of the Laplace equation can be written as Fourier power series $\phi_1 = {\phi_1}^0 + \sum_{n\neq 0}\phi_{1n}(t)(r/R)^{|n|}\mathrm{e}^{in\theta}$, $\phi_2 = {\phi_2}^0 + \sum_{n\neq 0}\phi_{2n}(t)(R/r)^{|n|}\mathrm{e}^{in\theta}$, where $\phi_i^0 = -(Q/2\pi)log(r/R) + C_i$ and C_i is the constant independent of both r and θ $(i = 1, 2)$. Similarly, we also assume that the perturbation of the interface can be represented as $\zeta(\theta, t) = \sum_{n=-\infty}^{\infty}\zeta_n(t)\mathrm{e}^{in\theta}$. From the condition that the area $S = \pi R(t)^2$ of fluid 1 is conserved independent of the perturbation $\zeta(\theta, t)$, there exist a constraint for $\zeta_0(t)$ such that $\zeta_0(t) = -(1/R(t))\sum_{n\neq 0}|\zeta_n(t)|^2$.

Then we consider two boundary conditions applied to the Laplace equation. The first one is known as kinematic boundary conditions $V_n = v_i \cdot \mathbf{n}$, which can be written in the polar coordinates as follows [18]:

$$\frac{\partial \mathscr{R}}{\partial t} = \left[\frac{1}{r^2}\frac{\partial \mathscr{R}}{\partial \theta}\frac{\partial \phi_i}{\partial \theta} - \frac{\partial \phi_i}{\partial r}\right]_{r=\mathscr{R}} (i = 1, 2). \tag{1}$$

Here $\mathbf{n}$ denotes the unit normal vector pointing into the interior of fluid 2. As the second boundary condition, we consider the balance of normal stresses at the interface,

$$\mathbf{n}\mathbf{T}_2\,\mathbf{n} - \mathbf{n}\mathbf{T}_1\,\mathbf{n} = \sigma(2/b + H), \tag{2}$$

instead of the traditional Young–Laplace equation. Here $\mathbf{T}_i = -p_i\mathbf{I} + 2\mu_i\mathbf{e}_i$ $(i = 1, 2)$ are the stress tensors for Newtonian fluids, with $\mathbf{I}$ as the unit tensor and $\mathbf{e}_i$ as the rate-of-strain tensor whose (j, k)-component $(\mathbf{e}_i)_{jk}$ is $(\mathbf{e}_i)_{jk} = (1/2)(\partial(v_i)_j/\partial x_k + \partial(v_i)_k/\partial x_j)$ $(j, k = 1, 2)$ and the index j of $(\cdot)_j$ means the j-component with respect to the two dimensional Cartesian coordinates (x_1, x_2), respectively [26,27]. Then the normal stress balance can be written simply as

$$p_1 - p_2 + [2\mu_1\mathbf{n}\cdot\mathbf{e}_1\cdot\mathbf{n} - 2\mu_2\mathbf{n}\cdot\mathbf{e}_2\cdot\mathbf{n}]_{r=\mathscr{R}} = \sigma\left(\frac{2}{b} + H\right). \tag{3}$$

The terms in $[\cdot]$ are referred to as the viscous normal stress ([23], hereafter VNS) and have not been considered in almost all previous studies based on Young–Laplace equation

$$p_1 - p_2 = \sigma\left(\frac{2}{b} + H\right). \tag{4}$$

Note that the Young–Laplace equation is easily derived from the normal stress balance by neglecting the VNS terms, which is satisfied in the case of $\mathbf{e}_i = 0$, *i.e.*, the interface moves rigidly without any deformation. This indicates that VNS plays an essential role to the deformation of the interface, so that we should employ not the Young–Laplace equation but the normal stress balance for the viscous fingering as a nonlinear phenomenon.

Now we derive the mode coupling equation. Substituting the general solution ϕ_i into the kinematic boundary condition (1), and eliminating ϕ_{in} by use of the normal stress balance (Eq. (3)), we obtain a time evolution equation for perturbation $\zeta(\theta, t)$,

$$\frac{\partial \zeta_n}{\partial t} \equiv \dot{\zeta}_n = \Lambda(n)\zeta_n + \sum_{n'\neq 0}\Gamma\left(n, n'\right)\zeta_{n'}\zeta_{n-n'}, \tag{5}$$

where

$$\begin{aligned}\Lambda(n) = {} & \frac{Q}{2\pi R^2}(A|n| - 1) - \frac{\alpha}{R^3}|n|\left(n^2 - 1\right) \\ & + \epsilon n^2\frac{Q}{\pi R^2}\left(A|n| - A^2 - 2\right) + 2\epsilon\frac{\alpha}{R^3}n^2\left(n^2 - 1\right)(A + |n|),\end{aligned} \tag{6}$$

$$\begin{aligned}\Gamma\left(n, n'\right) = {} & \frac{1}{R}\left[\frac{Q}{2\pi R^2}\left(A|n|\left\{A\left|n'\right|\left(1 - \mathrm{sgn}\left(nn'\right)\right) - \frac{1}{2}\right\} - \left(A\left|n'\right| - 1\right)\right)\right. \\ & \left. - \frac{\alpha}{R^3}\left(|n|\left(1 - \frac{1}{2}nn' - \frac{3}{2}n'^2\right) + \left\{A|n|\left(1 - \mathrm{sgn}\left(nn'\right)\right) - 1\right\}\left|n'\right|\left(n'^2 - 1\right)\right)\right] \\ & + \frac{\epsilon}{R} - \frac{Q}{\pi R^2}An^2\left|n'\right|\left(1 - A^2\right)\mathrm{sgn}\left(nn'\right) + A|n|\left\{-4n'^2\left(n' + 1\right)\right. \\ & \left. + \frac{1}{2}|n|\left(A + |n| + 4n'\right) + 1\right\} \\ & + A|n|\left(A\left|n'\right| - 1\right)\left\{|n|(A + \|n\|) + \left|n'\right|\left(2n'^2 - nn' + 2\right)\right\} \\ & - n'^2\left(A\left|n'\right| - A^2 + 2\right)\left\{A|n|\left(1 - \mathrm{sgn}\left(nn'\right)\right) - 1\right\} \\ & + 2\frac{\alpha}{R^3}n^2(A + |n|)\left(1 - \frac{1}{2}nn' - \frac{3}{2}n'^2\right) + n^2\left|n'\right|\left(n'^2 - 1\right)\left(1 - A^2\right)\mathrm{sgn}\left(nn'\right) \\ & + A|n|\left|n'\right|\left(n'^2 - 1\right)\left\{|n|(A + |n|) + \left|n'\right|\left(2n'^2 - nn' + 2\right)\right\} \\ & - |n|n'\left(n'^2 - 1\right)\left(4n' - n\right) + n'^2\left(n'^2 - 1\right)\left(A + \left|n'\right|\right)\left\{A|n|\left(1 - \mathrm{sgn}\left(nn'\right)\right) - 1\right\}.\end{aligned} \tag{7}$$

In the above, parameters are defined as $A = (\mu_2 - \mu_1)/(\mu_2 + \mu_1)$ and $\alpha = b^2\sigma/12(\mu_1 + \mu_2)$. Moreover $\epsilon = \epsilon(t) = b^2/12R(t)^2$ represents the effect of VNS and also indicates the 3-dimensional effect of the interface as seen from its definition. This equation, which was also derived in [25], reflects the VNS effect in the complete form, and therefore it is the extended one which proposed by Miranda and coauthors [18,24]. We also emphasize that the analysis by Kim et al. [23] was carried out based on the Eq. (5) for the case of $\Gamma(n, n') = 0$. For these reasons, we refer to Eq. (5) as the extended mode coupling equation [25].

3. Analysis of the extended mode coupling equation

3.1. Linear approximated solution

In this section we analyze the extended mode coupling equation (5) derived above. Following [18,24], as the first step, we neglect the coupling term of the Eq. (5) and solve the linearized equation,

$$\dot{\zeta}_n = \Lambda(n)\zeta_n. \tag{8}$$

The solution of Eq. (8) is easily obtained as

$$\zeta_n(t) = \zeta_n(0)exp\left(\int_0^t \Lambda(n)\mathrm{d}t\right). \tag{9}$$

This solution grows with t if $\Lambda(n) > 0$ and decays otherwise. The former corresponds to the unstable interface, and the latter means the

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