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A darkness theorem for the beam shape coefficients and its relationship to higher-order non-vortex Bessel beams

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ABSTRACT

Within the framework of generalized Lorenz–Mie theories and other light scattering theories such as the Extended Boundary Condition Method, the illuminating electromagnetic beam is described in terms of beam shape coefficients. We establish a darkness theorem in terms of the shape coefficients, allowing one to establish whether the beam intensity is zero on an axis (i.e. dark) or not. This theorem allows one to predict the existence of higher-order nonvortex Bessel beams. A proposal for similar studies concerning other types of beams is provided, as a possible extension of the present work.

1. Introduction

The description of illuminating beams in generalized Lorenz–Mie theories (GLMTs) [1] and other light scattering theories such as the Extended Boundary Condition Method (EBCM) [2], relies on an encoding of the beam under study in terms of coefficients called Beam Shape Coefficients (BSCs). About twenty years ago, a paper was devoted to a discussion of partial wave expansions and properties of axisymmetric beams, relying on the Poynting vector expressed in terms of BSCs [3]. That study, however, excluded the case of a class of beams, hereafter called dark beams, otherwise also called hollow beams [4], in which the energy flow in the direction of propagation along an axis, more specifically the beam axis, is zero. Many kinds of beams have been the subject of a large number of investigations during the last two decades. These include vortex beams which are dark beams, such as higher-order Bessel beams, higher-order Laguerre–Gaussian beams [5–11], which may be decomposed in terms of Hermite–Gaussian modes [5,6] , and higher-order Mathieu beams [4]. This paper provides a complement to [3] by focusing on dark beams.

The body of this study is organized as follows. Section 2 establishes a Darkness Theorem (DT) expressed in terms of BSCs, and corollary statements. In Section 3, this DT is used to comment on higher-order Bessel beams and the (somewhat unexpected) existence of higher-order Bessel beams that are not dark and thus have components that do not exhibit any phase singularity on the axis. Section 4 is a conclusion.

2. The darkness theorem

We consider a monochromatic electromagnetic beam with the time dependence $\exp(i\omega t)$ which is hereafter omitted, as is the normal practice. Let (x, y, z) be a Cartesian coordinate system and (r, θ, φ) the associated spherical coordinate system according to the standard definition. Let us assume that the beam is propagating along the z-direction (usually from the negative z's to the positive z's). The electric and magnetic field components along the r-, θ- and φ-directions are obtained by adding together TM- and TE-polarized solutions as in [12], and Section 3.5, pp. 51-52 of [1]:

$$E_r = kE_0 \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} g_{n,TM}^m e^{im\varphi} [\psi_n''(kr) + \psi_n(kr)] P_n^{|m|}(\cos\theta) \tag{1}$$

$$E_\theta = \frac{E_0}{r} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} e^{im\varphi} [g_{n,TM}^m \psi_n'(kr) \tau_n^{|m|}(\cos\theta) + m g_{n,TE}^m \psi_n(kr) \pi_n^{|m|}(\cos\theta)] \tag{2}$$

$$E_\varphi = \frac{iE_0}{r} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} e^{im\varphi} [m g_{n,TM}^m \psi_n'(kr) \pi_n^{|m|}(\cos\theta) + g_{n,TE}^m \psi_n(kr) \tau_n^{|m|}(\cos\theta)] \tag{3}$$

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$$H_r = kH_0 \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} g_{n,TE}^m e^{im\varphi} [\psi_n''(kr) + \psi_n(kr)] P_n^{|m|}(\cos\theta) \quad (4)$$

$$H_\theta = \frac{H_0}{r} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} e^{im\varphi} [g_{n,TE}^m \psi_n'(kr) \tau_n^{|m|}(\cos\theta) - m g_{n,TM}^m \psi_n(kr) \pi_n^{|m|}(\cos\theta)] \quad (5)$$

$$H_\varphi = \frac{iH_0}{r} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} c_n^{pw} e^{im\varphi} [m g_{n,TE}^m \psi_n'(kr) \pi_n^{|m|}(\cos\theta) - g_{n,TM}^m \psi_n(kr) \tau_n^{|m|}(\cos\theta)], \quad (6)$$

in which E_0 and H_0 are electric and magnetic field strengths respectively, ψ_n denotes Riccati–Bessel functions with the argument kr (k the wavenumber), a prime denotes a derivative of a function with respect to its argument, the coefficients c_n^{pw} ("pw" standing for "plane wave") are coefficients which occur in a natural way in the Bromwich formulation of Lorenz–Mie theory [13]:

$$c_n^{pw} = \frac{1}{ik} (-i)^n \frac{2n+1}{n(n+1)}. \quad (7)$$

Also, $g_{n,TM}^m$ and $g_{n,TE}^m$ are the usual TM- and TE-BSCs of GLMT, while τ_n^m and π_n^m, with argument $\cos\theta$, are generalized Legendre functions defined according to:

$$\pi_n^m(\cos\theta) = \frac{P_n^m(\cos\theta)}{\sin\theta} \quad (8)$$

$$\tau_n^m(\cos\theta) = \frac{dP_n^m(\cos\theta)}{d\theta}, \quad (9)$$

in which $P_n^m(\cos\theta)$ are the associated Legendre functions defined according to Hobson's convention:

$$P_n^m(\cos\theta) = (-1)^m (\sin\theta)^m \frac{d^m P_n(\cos\theta)}{(d\cos\theta)^m}, \quad (10)$$

in which $P_n(\cos\theta)$ are the Legendre polynomials.

We now focus our interest on the behavior of the fields on the positive z-axis, $\theta = 0$. From the definitions of the generalized Legendre functions, we readily establish that:

$$k\pi_n^k(1) = \tau_n^k(1) = 0, \; k \neq 1 \quad (11)$$

$$\pi_n^1(1) = \tau_n^1(1) = \Omega_n \neq 0 \quad (12)$$

which defines the quantity Ω_n. Furthermore, from the definition of the associated Legendre functions, we have:

$$P_n^{|m|}(1) = P_n(1)\delta_{|m|0} = \Delta_n \delta_{|m|0} \quad (13)$$

which defines the quantity Δ_n. The quantities Ω_n and Δ_n are $(-n)(n+1)/2$ and 1, respectively. The symbols Ω_n and Δ_n are however retained in the formula to obtain a better view of the involved symmetries.

Therefore, when expressing the field components of Eqs. (1)–(6) for $\theta = 0$, only the modes $m = \pm 1$ have to be retained for the angular components, and $m = 0$ for the radial components, leading to:

$$(E_r)_{\theta=0} = kE_0 \sum_{n=1}^{\infty} c_n^{pw} g_{n,TM}^0 [\psi_n''(kr) + \psi_n(kr)] \Delta_n \quad (14)$$

$$(E_\theta)_{\theta=0} = \frac{E_0}{r} e^{i\varphi} \sum_{n=1}^{\infty} c_n^{pw} \Omega_n [g_{n,TM}^1 \psi_n'(kr) + g_{n,TE}^1 \psi_n(kr)] + \frac{E_0}{r} e^{-i\varphi} \sum_{n=1}^{\infty} c_n^{pw} \Omega_n [g_{n,TM}^{-1} \psi_n'(kr) - g_{n,TE}^{-1} \psi_n(kr)] \quad (15)$$

$$(E_\varphi)_{\theta=0} = \frac{iE_0}{r} e^{i\varphi} \sum_{n=1}^{\infty} c_n^{pw} \Omega_n [g_{n,TM}^1 \psi_n'(kr) + g_{n,TE}^1 \psi_n(kr)] + \frac{iE_0}{r} e^{-i\varphi} \sum_{n=1}^{\infty} c_n^{pw} \Omega_n [-g_{n,TM}^{-1} \psi_n'(kr) + g_{n,TE}^{-1} \psi_n(kr)] \quad (16)$$

$$(H_r)_{\theta=0} = kH_0 \sum_{n=1}^{\infty} c_n^{pw} g_{n,TE}^0 [\psi_n''(kr) + \psi_n(kr)] \Delta_n \quad (17)$$

$$(H_\theta)_{\theta=0} = \frac{H_0}{r} e^{i\varphi} \sum_{n=1}^{\infty} c_n^{pw} \Omega_n [g_{n,TE}^1 \psi_n'(kr) - g_{n,TM}^1 \psi_n(kr)] + \frac{H_0}{r} e^{-i\varphi} \sum_{n=1}^{\infty} c_n^{pw} \Omega_n [g_{n,TE}^{-1} \psi_n'(kr) + g_{n,TM}^{-1} \psi_n(kr)] \quad (18)$$

$$(H_\varphi)_{\theta=0} = \frac{iH_0}{r} e^{i\varphi} \sum_{n=1}^{\infty} c_n^{pw} \Omega_n [g_{n,TE}^1 \psi_n'(kr) - g_{n,TM}^1 \psi_n(kr)] + \frac{iH_0}{r} e^{-i\varphi} \sum_{n=1}^{\infty} c_n^{pw} \Omega_n [-g_{n,TE}^{-1} \psi_n'(kr) - g_{n,TM}^{-1} \psi_n(kr)]. \quad (19)$$

Let us remark that the φ-dependence of the radial components has been cancelled away, while the angular components still exhibit φ-dependence of the form $\exp(i\varphi)$ and $\exp(-i\varphi)$. Now, let us consider a vector $\mathbf{V}$ expressed in terms of Cartesian and spherical coordinates:

$$V_x = V_r \sin\theta \cos\varphi + V_\theta \cos\theta \cos\varphi - V_\varphi \sin\varphi \quad (20)$$

$$V_y = V_r \sin\theta \sin\varphi + V_\theta \cos\theta \sin\varphi + V_\varphi \cos\varphi \quad (21)$$

$$V_z = V_r \cos\theta - V_\theta \sin\theta. \quad (22)$$

Hence, "on the axis", for $\theta = 0$:

$$(V_x)_{\theta=0} = (V_\theta)_{\theta=0} \cos\varphi - (V_\varphi)_{\theta=0} \sin\varphi \quad (23)$$

$$(V_y)_{\theta=0} = (V_\theta)_{\theta=0} \sin\varphi + (V_\varphi)_{\theta=0} \cos\varphi \quad (24)$$

$$(V_z)_{\theta=0} = (V_r)_{\theta=0}. \quad (25)$$

Inserting Eqs. (14)–(19) into Eqs. (23)–(25), we obtain the Cartesian components in which the azimuthal angle φ no longer occurs:

$$(E_x)_{\theta=0} = \frac{E_0}{r} \sum_{n=1}^{\infty} c_n^{pw} \Omega_n [(g_{n,TM}^1 + g_{n,TM}^{-1}) \psi_n'(kr) + (g_{n,TE}^1 - g_{n,TE}^{-1}) \psi_n(kr)] \quad (26)$$

$$(E_y)_{\theta=0} = \frac{iE_0}{r} \sum_{n=1}^{\infty} c_n^{pw} \Omega_n [(g_{n,TM}^1 - g_{n,TM}^{-1}) \psi_n'(kr) + (g_{n,TE}^1 + g_{n,TE}^{-1}) \psi_n(kr)] \quad (27)$$

$$(E_z)_{\theta=0} = kE_0 \sum_{n=1}^{\infty} c_n^{pw} g_{n,TM}^0 [\psi_n''(kr) + \psi_n(kr)] \Delta_n \quad (28)$$

$$(H_x)_{\theta=0} = \frac{H_0}{r} \sum_{n=1}^{\infty} c_n^{pw} \Omega_n [(g_{n,TE}^1 + g_{n,TE}^{-1}) \psi_n'(kr) + (g_{n,TM}^{-1} - g_{n,TM}^1) \psi_n(kr)] \quad (29)$$