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Accounting for Gaussian quadrature in four-stream radiative transfer algorithms

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ABSTRACT

The Gaussian–Legendre, Gaussian–Lobatto, Gaussian–Chebyshev and Gaussian quadrature (GQ) with different moment powers have been investigated by applying them into the four-stream solar and infrared radiative transfer algorithms. For solar radiative transfer, the Gaussian–Chebyshev and GQ with moment power $m=0$ show relatively accurate results compared to other types of 2GQ in a single-layer scattering medium. In a real atmospheric profile including gaseous transmission, Gaussian–Chebyshev and GQ with moment power $m=0$ are comparable in accuracy for cloud heating rate. GQ with moment power $m=0$ produces more accurate results in the upward flux at the top of the atmosphere, while Gaussian–Chebyshev produces more accurate results in the downward flux at the surface. These results have been confirmed in evaluations by using satellite observation data. For infrared radiative transfer, the GQ with moment powers $m = 0, 2, 4$ show relatively accurate results in effective emissivity for a single-layer scattering medium. In a real atmospheric profile, the GQ with moment powers $m=0$ and $m=2$ show superior accuracy in heating rate and flux. In addition, the evaluations using satellite observation data also show that the accuracy of GQ with moment powers $m=0$ and $m=2$ is comparable. Both the schemes are the best candidates for the four-stream radiation algorithms.

1. Introduction

Gaussian quadrature (GQ) is an accurate and effective method to deal with the definite integral of a function, which is usually illustrated as a weighted sum at specified points of function values. GQ was first time applied to solve radiative transfer equations (RTE) by Chandrasekhar [1] and consequently a method called discrete ordinates method (DOM) was created. In RTE, the integral term to deal with the scattering effect can often be decomposed by using GQ [2–4].

The physical process of radiative transfer is described by a differential–integral equation. The integral part represents the multiple-scattering into a certain direction. In order to solve the radiative transfer equation, the differential–integral equation has to be converted to a pure differential equation. In the DOM, the integral part can be decomposed and resolved by GQ.

In solar radiation, the two-stream approximation [5], which corresponding to 1-node GQ, has been widely used in current climate models [6]. However, the cloud heating from the two-stream approximation might have been underestimated by as much as 10% [7–10]. Therefore, the four-stream approximation [3,11–17], which corresponds to a 2-node GQ in the integral intervals of $[-1\ 0]$ and $[0\ 1]$ respectively, is developed. The single-layer analytical solutions of the four-stream approximations have been found [3,12,14,16]. Recently, Zhang et al. [9] have derived an adding method for four-stream discrete ordinates method (DOM) for solar radiation, referred as 4DDA, which is based on Chandrasekhar's invariance principle [1].

In infrared radiation, the scattering effect is very weak. In most current climate models, a method of absorption approximation (AA) is widely used [6], in which the scattering process is neglected. The diffuse transmission in AA can be calculated by using a diffusivity factor of 1.66 [18,15]. Li [19] has approved that the accurate diffusivity factor should be $\sqrt{e} = 1.648721$. The proof was based on the GQ integration of moments. Studies have shown that the AA scheme can cause an overestimation of outgoing longwave radiation in climate simulations with errors up 4 W/m^2 [20,21,15]. In order to include the infrared scattering effect, the two-stream approximation has been applied to the infrared radiation [21]. Compared to AA, the two-stream method is more accurate, however, as shown in [20,15], the relative errors of emissivities by using two-stream method can be over 10% under a thin optical

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thickness condition. This indicates that a higher node Gaussian quadrature is needed to improve the accuracy. The four-stream analytical solution to the infrared RTE has been found [22]. Based on Chandrasekhar's invariance principle [1], Zhang et al. [23] have derived 4DDA to deal with layer connections in the infrared radiative transfer.

There are many types of GQ. In early times, the Gauss–Legendre quadrature was mainly used for RTE in the atmospheric radiation community [1–3]. Later, the so-called double GQ was introduced [4,22]. It is shown that the double GQ can produce more accurate results compared to Gauss–Legendre quadrature. This indicates that the GQ scheme plays a key role in radiative transfer process. When a discrete ordinates type of solution to RTE is found, the accuracy relies only on the choice of GQ schemes. However, other types of GQ such as the Gauss–Chebyshev quadrature, Gauss–Lobatto quadrature, Gaussian integration of moments, etc., have rarely been applied in RTE. Most types of GQ have an integral interval of $[-1, 1]$, which matches with the scattering integral interval of RTE. The application of GQ to RTE is straightforward. However, the Gaussian integration of moments cannot be directly applied to solve RTE because its integral interval is $[0, 1]$ [24]. In order to apply it to RTE, the integral interval for Gaussian integration of moments has to be extended. The so-called double GQ is actually a specific example of the Gaussian integration of moments with moment equal to zero. The Gaussian integration of a higher moment has not been investigated yet.

Given that there are various types of GQ, this study mainly focuses on the comparison of different types of GQ in the solar and infrared RTE. We will focus on the schemes of 2-node GQ (denoted as 2GQ), which match with the four-stream radiative transfer discrete ordinates solution. The four-stream radiative transfer algorithm will become more and more popular in climate models, as computing speed increases. In the following Section 2, several types of Gaussian integration are discussed, and the integral interval for Gaussian quadrature of moments is extended. In Section 3, different types of GQ are applied to the solar radiative transfer. The accuracy of four-stream approximation by using different types of 2GQ is systematically investigated by comparing with 128-stream discrete ordinates calculations. In Section 4, different types of GQ are applied to the infrared radiative transfer. The accuracy of four-stream approximation by using different types of 2GQ is systematically investigated by comparing with δ-128-stream discrete ordinates scheme (δ-128S). In the final section, a summary is given.

2. Different types of Gaussian quadrature

2.1. Gauss–Legendre quadrature

The Gauss–Legendre quadrature states [24]:

$$\int_{-1}^{1} f(\mu)d\mu = \sum_{i=-n}^{n} a_i f(\mu_i), \tag{1}$$

where μ_i is the abscissas and a_i is the weight. μ_i is symmetrical in the intervals $[-1, 0]$ and $[0, 1]$ with $\mu_{-i} = -\mu_i$. For 2-node Gaussian-Legendre quadrature, referred as 2GQ(Legendre), the values of μ_i and a_i are listed in Table 1.

2.2. Gauss–Chebychev quadrature

The Gauss–Chebychev quadrature states [25]:

$$\int_{-1}^{1} f(\mu)d\mu = \sum_{i=-n}^{n} a_i f(\mu_i), \tag{2}$$

Table 1
Some types of abscissa and weight factors for 2-node Gaussian integration in the paper.

Gaussian integration	μ_1,μ_2	a_1,a_2
Legendre	0.3399810	0.6521452
	0.8611363	0.3478548
Chebychev	0.7946545	0.5000000
	0.1875925	0.5000000
Lobatto	1.0000000	0.1666667
	0.4472136	0.8333333
$m=0$	0.2113248	0.5000000
	0.7886751	0.5000000
$m=2$	0.0947241	0.3023577
	0.6756462	0.6976424
$m=4$	0.0693003	0.2454125
	0.6366946	0.7545875
$m\to\infty$	0.0329023	0.1464466
	0.5566679	0.8535534

where μ_i is the abscissas and $a_i = \frac{1}{n}$. For 2-node Gaussian–Chebychev quadrature, referred as 2GQ(Chebychev), the values of μ_i and a_i are listed in Table 1.

2.3. Gauss–Lobatto quadrature

There is an interesting characteristic for Gauss–Lobatto quadrature with abscissa $\mu_{-n} = -1$ and $\mu_n = 1$ for any values of n [26]. The Gauss–Lobatto quadrature states:

$$\int_{-1}^{1} f(\mu)d\mu = a_{-n}f(-1) + \sum_{i=-(n-1)}^{n-1} a_i f(\mu_i) + a_n f(1), \tag{3}$$

where μ_i is the abscissas and a_i is the weight. For 2-node Gauss–Lobatto, referred as 2GQ(Lobatto), the values of μ_i and a_i are listed in Table 1.

2.4. Gaussian integration of moments

By n-node GQ, the integration of moment m is evaluated by [24]

$$\int_{0}^{1} x^m f(x)dx = \sum_{i=1}^{n} w_i f(x_i), \tag{4}$$

where x_i is the abscissas; w_i is the weight. In (4), the values of x_i and w_i for $m \le 7$ are shown in [24]. For higher moments of $m > 7$, x_i and w_i are calculated by Li [19]. In [19], the high moment result has even been extended to $m \to \infty$.

The Gaussian integration of moments cannot be applied to solve the radiative transfer equation directly due to the limitation of integral interval. By substitution of $\mu = x^{m+1}$,

$$\begin{aligned}\int_{0}^{1} f(\mu)d\mu &= (m+1)\int_{0}^{1} x^m f(\mu)dx \\ &= (m+1)\sum_{i=1}^{n} w_i f(x^{m+1}) \\ &= \sum_{i=1}^{n} a_i f(\mu_i),\end{aligned} \tag{5}$$

where

$$\mu_i = x_i^{m+1} \quad \text{and} \quad a_i = (m+1)w_i. \tag{6}$$

For even number of $m = 0, 2, 4, \ldots$, by substituting $\mu = (-x)^{m+1}$, we obtain