Journal of Quantitative Spectroscopy & Radiative Transfer 196 (2017) 17–45

Contents lists available at ScienceDirect

Journal of Quantitative Spectroscopy & Radiative Transfer

journal homepage: www.elsevier.com/locate/jqsrt

A review of the matrix-exponential formalism in radiative transfer

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A R T I C L E I N F O

Article history:
Received 24 February 2017
Accepted 24 February 2017
Available online 19 March 2017

Keywords:
Matrix-exponential
Discrete ordinate method
Matrix operator method
Matrix Riccati equations
Asymptotic theory

A B S T R A C T

This paper outlines the matrix exponential description of radiative transfer. The eigendecomposition method which serves as a basis for computing the matrix exponential and for representing the solution in a discrete ordinate setting is considered. The mathematical equivalence of the discrete ordinate method, the matrix operator method, and the matrix Riccati equations method is proved rigorously by means of the matrix exponential formalism. For optically thin layers, approximate solution methods relying on the Padé and Taylor series approximations to the matrix exponential, as well as on the matrix Riccati equations, are presented. For optically thick layers, the asymptotic theory with higher-order corrections is derived, and parameterizations of the asymptotic functions and constants for a water-cloud model with a Gamma size distribution are obtained.

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http://dx.doi.org/10.1016/j.jqsrt.2017.02.015
0022-4073/

1. Introduction

The radiative transfer is an important issue for astrophysics, atmospheric physics, meteorology and engineering sciences. A wide range of solution methods of the radiative transfer equation (RTE) have been proposed (see, e.g., [1–11] and references therein for a general review). The discrete ordinate method [12–14,6] and the matrix operator method [15–18] involve replacing the continuous dependence of the radiance on direction by a dependence on a discrete set of directions. For a homogeneous layer, the discretized radiative transfer equation then takes the form of a system of linear first-order differential equations. In the classical discrete ordinate method of Chandrasekhar, the solution of the system of equations is expressed as a linear combination of characteristic solutions of the discretized problem, while the matrix operator method is primarily oriented toward numerical computations of the reflection and transmission matrices. Another group of methods are based on the concept of invariant imbedding, which is due to Ambarzumian [19]. Ambarzumian derived an equation for the reflection function of a semi-infinite atmosphere by noting that the reflection function remains unchanged upon addition of a new layer. This technique was further generalized by Chandrasekhar [13] to a finite layer, while Bellman et al. [20] showed that the reflection function derived by using the invariant imbedding satisfies the Riccati equation.

The system of differential equations of the discretized radiative transfer equation can be solved by using a classical mathematical procedure involving the matrix exponential operator, in which the system matrix appears in the exponent. Waterman [21] was the first who provided a matrix exponential description of radiative transfer. Mathematical elegance aside, he showed its practical value in radiative transfer computations from both the analytical and purely numerical point of view. In particular, Waterman related the matrix exponential to the extinction matrix incorporating the reflection and transmission matrices of a homogeneous layer, provided an eigenvector representation of the matrix exponential, derived analytical expressions for the reflection and transmission matrices in the limit of small and large optical thicknesses, showed that the matrix exponential can be used to generate starting values for the doubling method, and applied the matrix exponential formalism to conservative scattering. Flatau and Stephens [22] extended the concept of matrix exponential of a homogeneous layer to an inhomogeneous atmosphere by introducing the so-called propagator (matrix) operator. As Waterman, Flatau and Stephens related the propagator to the extinction matrix of a homogeneous layer, notified the similarity between the matrix exponential solution and Chandrasekhar's discrete ordinate solution, established various properties of the propagator and used them to derive the Riccati matrix equations for an inhomogeneous atmosphere, as well as the adding and doubling formulas. Although in both papers [21,22] an eigendecomposition method for computing the matrix exponential is considered, explicit and stable representations of the reflection and transmission matrices are not given. This problem has been solved by Nakajima and Tanaka [18] by using a system of characteristic solutions of the discretized problem, and by Budak et al. [23,24] by using the matrix exponential formalism. It should be also mentioned that Doicu and Trautmann [25,26] designed the so-called discrete ordinate method with matrix exponential to compute the radiance field in a multi-layered atmosphere.

The purpose of this paper is to provide a consistent overview of the matrix exponential description of radiative transfer. We mainly focus on a mathematical rigorous and self-contained analysis based on the results given in [21,22,27] and our own results [25,26,28,29]. The final goals are to prove the mathematical equivalence of the discrete ordinate method, matrix operator method, and the matrix Riccati equations method, on the one hand, and to derive efficient computation formulas for the reflection and transmission matrices in the limit of small and large optical thicknesses, on the other hand.

The rest of the paper is organized as follows. In Section 2, we present the discrete ordinate setting in which the matrix exponential method is applied, while in Section 3, we discuss the eigendecomposition method for computing the matrix exponential. Section 4 is devoted to the discrete ordinate method with matrix exponential. In Section 5, dealing with the matrix operator method with matrix exponential, we derive several representations of the reflection and transmission matrices for arbitrary optical thickness, as well as for small and large optical thicknesses. In Section 6 we establish the matrix Riccati equations, prove the mathematical equivalence between the matrix Riccati equations method and the matrix exponential method in computing the reflection and transmission matrices of a homogeneous layer, and discuss some approximation solution methods for small values of the optical thickness and/or single scattering albedo. Finally, Section 7 contains some concluding remarks. Additional results dealing with a justification of the Gaussian quadrature in the discrete ordinate method, a review of eigendecomposition methods for computing the matrix exponential, and an extension of the analytical results to conservative scattering are presented in appendices.

2. Matrix formulation of the radiative transfer equation

For a given solar direction $\mathbf{\Omega}_0 = (-\mu_0, \varphi_0)$, with $\mu_0 > 0$ being the cosine of the solar zenith angle and φ_0 the solar azimuthal angle, the equation describing the radiative transfer in a plane-parallel homogeneous layer of optical thickness $\bar{\tau}$ is

$$\mu\frac{\mathrm{d}I_\mathrm{d}(\tau, \mu, -\mu_0, \varphi-\varphi_0)}{\mathrm{d}\tau} = I_\mathrm{d}(\tau, \mu, -\mu_0, \varphi-\varphi_0) - \frac{\omega}{4\pi}F_0 p(\mu, -\mu_0, \varphi-\varphi_0)\mathrm{e}^{-\tau/\mu_0} - \frac{\omega}{4\pi}\int_0^{2\pi}\int_{-1}^{1} p(\mu, \mu', \varphi-\varphi')I_\mathrm{d}(\tau, \mu', -\mu_0\varphi'-\varphi_0)\mathrm{d}\mu'\mathrm{d}\varphi', \tag{1}$$

where $I_\mathrm{d}(\tau, \mu, -\mu_0, \varphi-\varphi_0)$ is the diffuse radiance at optical depth τ along the direction specified by the cosine of the zenith angle μ and the azimuthal angle φ, $p(\mu, \mu', \varphi-\varphi')$ is the scattering phase function for the radiation scattered from the direction $\mathbf{\Omega}' = (\mu', \varphi')$ into the direction $\mathbf{\Omega} = (\mu, \varphi)$, ω is the single scattering albedo, and F_0 is the solar flux. For simplicity, the thermal emission term is neglected in Eq. (1). If the homogeneous layer is placed in a multi-layered atmosphere at optical depth τ_0, the radiative transfer equation for the diffuse radiance $I_\mathrm{d}(\tau_0+\tau, \mu, -\mu_0, \varphi-\varphi_0)$ contains the direct transmission term $\exp[-(\tau_0+\tau)/\mu_0]$ instead of $\exp(-\tau/\mu_0)$. The total radiance, defined in terms of the diffuse and direct radiances by