Journal of Quantitative Spectroscopy & Radiative Transfer 185 (2016) 22–26

Contents lists available at ScienceDirect

Journal of Quantitative Spectroscopy & Radiative Transfer

journal homepage: www.elsevier.com/locate/jqsrt

Generalization of the Optical Theorem to the multipole source excitation

Yuri Eremin [a], Thomas Wriedt [b,*]

[a] Faculty of Computational Mathematics and Cybernetics, Lomonosov Moscow State University, Lenin's Hills, 119991 Moscow, Russia
[b] Institut für Werkstofftechnik, Badgasteiner Str. 3, 28359 Bremen, Germany

ARTICLE INFO

Article history:
Received 9 June 2016
Received in revised form
12 August 2016
Accepted 14 August 2016
Available online 20 August 2016

Keywords:
The Optical Theorem
Multipole source
Maxwell's theory
Gauss Theorem
Extinction cross-section

ABSTRACT

Based on classic Maxwell's theory and the Gauss Theorem we extended the Optical Theorem to the case of a penetrable particle excited by a multipole source. We demonstrate that the derived extinction cross-section can be evaluated via calculation of some specific derivatives from the scattered field at the point of the multipole location. The obtained relation between extinction cross-section and scattering cross-section can be employed to estimate the corresponding absorption cross-section.

1. Introduction

The Optical Theorem (OT) in electromagnetics has a long history; its occurrence in scattering theory starts more than one hundred years ago and similar results can also be found in acoustical scattering, seismics and quantum mechanics [1–3]. The term OT has been well known ever since [4]. The OT is a basic result in scattering theory, relating the extinction cross section of a structure to the scattering amplitude in the forward direction. Over the years, many derivations and implementations of the theorem have been provided [5–8]. In computational electromagnetics this theorem is employed for checking or verification of the results of light scattering models, since for a non absorbing particle, the total scattering cross-section must be proportional to the imaginary part of the forward scattering amplitude [9,10].

The theorem has been reconsidered and generalized by a number of researchers to consider plane wave scattering by a particle near an interface between media with different refractive indices [11,12], and electromagnetic wave propagation in anisotropic and bianisotropic media [13,14]. The OT was extended to the case of seismic wave propagation [15], as well as to rough surfaces and beam excitation [16,17]. Excitation by a local source requires a new approach, and the OT has up to now been extended to the point source excitation of a particle located in free space [18,19].

It is known, that a dipole radiation pattern is insufficient for directional scanning to obtain obstacle location area. This can be achieved by using multipoles, which allow to generate more directive emission diagrams. In the present paper we extended the OT to the case of a penetrable particle deposited in free space excited by an electric multipole of arbitrary order. We use classic Helmholtz – and Maxwell's theories and the Green Formulas and the Gauss Theorem as basic techniques. This generalization will enables to test scattering models in case of wave scattering by non-absorbing particles.

The paper is organized as follows. In the next section we consider the scalar case of wave excitation, which has some own interest due to the variety of application in

* Corresponding author. Fax: +49(0)42121853211.
E-mail address: thw@iwt.uni-bremen.de (T. Wriedt).

http://dx.doi.org/10.1016/j.jqsrt.2016.08.011
0022-4073/

acoustics, nondestructive testing, modeling in medicine and scattering problems connected to environmental data analysis. We performed a detailed analysis of the scalar scattering problem, which helps us to complete our consideration with the electromagnetic case. At the following section we will examine a penetrable particle excited by an electric multipole of an arbitrary order.

2. The Optical Theorem for the scalar case

We begin our consideration from the mathematical statement of the scattering problem for the scalar field generated by a multipole source deposited in free space $\mathfrak{R}^3$ at a point M_0 in the presence of a bounded penetrable particle D_i with a smooth surface $\partial D_i \in C^{(2,\alpha)}$ (HÖlder space). Then the mathematical statement of the scattering problem can be presented as

$$\begin{aligned}
&\Delta U_0 + k_0^2 U_0 = -J(M, M_0), \quad M_0 \in D_0 := R^3/\overline{D_i};\\
&\Delta U_i + k_i^2 U_i = 0, \quad M \in D_i;\\
&[U(P)] = [\partial U(P)/\partial n] = 0, \quad P \in \partial D_i;\\
&\frac{\partial U_0}{\partial r} + jk_0 U_0 = o(1/r), \quad r := |M| \to \infty.
\end{aligned} \tag{1}$$

where $J(M, M_0)$ is defined by the particular form and order of the multipole, which will be specified later; [.] denotes the jump of the fields crossing the surface ∂D_i, n is the unit normal to the particle surface ∂D_i, and $k_i^2 = const, \mathrm{Im} k_i^2 \le 0$ which corresponds to the time dependence $\exp\{j\omega t\}$.

We choose the origin of a Cartesian coordinate system and direct its Oz axis so that it passes through the point $M_0 = (0, 0, z_0)$ corresponding to the source position. Let us introduce the multipoles, which in spherical coordinate system accept the form

$$\begin{aligned}
&w_n^m(M, M_0) := h_n^{(2)}(k_0 r) P_n^m(\cos\theta) e^{-jm\varphi}; \quad n = 0, 1, \ldots;\\
&m = 0, \pm 1, \ldots; \quad |m| \le n
\end{aligned} \tag{2}$$

For the multipoles (2) the following fundamental representation is valid [20]

$$\begin{aligned}
&h_n^{(2)}(k_0 R_{MM_0}) P_n^m(\cos\theta) e^{-jm\varphi}\\
&= (-1)^m j^n \left[\frac{j}{k_0}\left(\frac{\partial}{\partial x} - j\frac{\partial}{\partial y}\right)\right]^m P_n^{(m)}\left(\frac{j}{k_0}\frac{\partial}{\partial z}\right) h_0^{(2)}(k_0 R_{MM_0})
\end{aligned}$$

here $h_n^{(2)}(x)$ is a spherical Hankel function, $R_{MM_0} = |M - M_0|$, $P_n^{(m)}(\cos\theta) = \frac{d^m P_n(\cos\theta)}{d(\cos\theta)^m}$, $P_n(\cos\theta)$ is a Legendre polynomial. Taking into account that the fundamental solution to the Helmholtz equation $\Psi(M, M_0)$: $\Delta\Psi + k_0^2\Psi = -\delta(M - M_0)$ has the form

$$\Psi(M, M_0) = \frac{e^{-jk_0 R_{MM_0}}}{4\pi R_{MM_0}} = -j\tfrac{4\pi}{k_0} h_0^{(2)}(k_0 R_{MM_0}), \text{ then}$$

$$\begin{aligned}
\Delta w_n^m + k_0^2 w_n^m &= \left(\Delta + k_0^2\right)(-1)^m j^n \left[\frac{j}{k_0}\left(\frac{\partial}{\partial x} - j\frac{\partial}{\partial y}\right)\right]^m\\
&\times P_n^{(m)}\left(\frac{j}{k_0}\frac{\partial}{\partial z}\right) h_0^{(2)}(k_0 R_{MM_0}) = -\frac{4\pi}{k_0} D_n^m \delta(M - M_0)
\end{aligned}$$

where δ is the Dirac delta function, and differential operator D_n^m is defined as

$$D_n^m = (-1)^m j^{n-1} \left[\frac{j}{k_0}\left(\frac{\partial}{\partial x} - j\frac{\partial}{\partial y}\right)\right]^m P_n^{(m)}\left(\frac{j}{k_0}\frac{\partial}{\partial z}\right) \tag{3}$$

So, returning to (1) it is easy to conclude that

$$J(M, M_0) = \frac{4\pi}{k_0} D_n^m \delta(M - M_0) \tag{4}$$

In this case, the scattering problem (1) has a unique solution [21].

Let us apply the Green Theorem [21] to the scattering problem (1) solutionU_0. Choose a sphere D_R of radius R, so that M_0 and D_i are inside the sphere and the D_R boundary will be referred as Σ_R. Applying the second Green's formula inside D_R/D_i to U_0 and the complex conjugate U_0^* we get

$$\int_{D_R/\overline{D_i}} (\Delta U_0 \cdot U_0^* - \Delta U_0^* \cdot U_0) d\tau = \int_{\Sigma_R \cup \partial D_i} \left\{\frac{\partial U_0}{\partial n} U_0^* - \frac{\partial U_0^*}{\partial n} U_0\right\} d\sigma \tag{5}$$

Here $\partial/\partial n$ is the normal derivative to the corresponding surface directed outside of the region D_R/D_i. Then, the left-hand side of (5) can be transformed to

$$\int_{D_R/\overline{D_i}} (\Delta U_0 \cdot U_0^* - \Delta U_0^* \cdot U_0) d\tau = 2j\mathrm{Im} \int_{D_R/\overline{D_i}} J^*(P, M_0) U_0(P) d\tau_P \tag{6}$$

Therefore, the relation (5) can be rewritten in the following form:

$$\mathrm{Im} \int_{D_R/\overline{D_i}} J^*(P, M_0) U_0(P) d\tau_P = \mathrm{Im} \int_{\Sigma_R} \frac{\partial U_0}{\partial n} U_0^* d\sigma - \mathrm{Im} \int_{\partial D_i} \frac{\partial U_0}{\partial n^+} U_0^* d\sigma \tag{7}$$

where $\partial/\partial n^+$ is an external normal derivative with respect to D_i . Employing the properties of the δ function [22], the integral in the left part of (7) can be transformed to

$$\begin{aligned}
&\mathrm{Im} \int_{D_R/\overline{D_i}} J^*(P, M_0) U_0(P) d\tau_P\\
&= \mathrm{Im}\left\{\frac{4\pi}{k_0} \int_{D_R/\overline{D_i}} \left[D_n^{m*} \delta(P - M_0)\right] U_0(P) d\tau_P\right\}\\
&= \frac{4\pi}{k_0} \mathrm{Im} \int_{D_R/\overline{D_i}} \left[\left(D_n^{m*}\right)^T U_0(P)\right] \delta(P - M_0) d\tau_P
\end{aligned} \tag{8}$$

here $D_n^{m*} = (-j)^{n-1}\left[\frac{j}{k_0}\left(\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right)\right]^m P_n^{(m)}\left(-\frac{j}{k_0}\frac{\partial}{\partial z}\right)$ is a complex conjugated operator and $\left(D_n^{m*}\right)^T$ is transposed with respect to D_n^{m*}. So, the operator $\left(D_n^{m*}\right)^T$ is as follows:

$$\left(D_n^{m*}\right)^T = (-1)^{m+n-1} j^{n-1} \left[\frac{j}{k_0}\left(\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right)\right]^m P_n^{(m)}\left(\frac{j}{k_0}\frac{\partial}{\partial z}\right) \tag{9}$$

Consider the first integral in the right part of (7). Employing the Sommerfeld radiation condition (1) we get

$$\mathrm{Im} \lim_{R\to\infty} \int_{\Sigma_R} \frac{\partial U_0}{\partial r} U_0^* d\sigma = -k_0 \lim_{R\to\infty} \int_{\Sigma_R} |U_0|^2 d\sigma_r$$

Using the definition of the Far Field Pattern $F(\theta, \varphi)$ [21]

$$U_0(M) = \frac{e^{-jk_0 r}}{k_0 r} F(\theta, \varphi) + o(1/r), \quad r \to \infty$$

the integral accepts the form

$$\mathrm{Im} \lim_{R\to\infty} \int_{\Sigma_R} \frac{\partial U_0}{\partial r} U_0^* d\sigma = -\frac{1}{k_0} \int_{\Omega} |F|^2 d\omega \tag{10}$$

where $\Omega = \{0 \le \theta \le \pi; \quad 0 \le \varphi \le 2\pi\}$ is a unit sphere.