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Bilinear Expansion For Redistribution Functions

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ABSTRACT

We suggest here a method for construction of a bilinear expansion for an angle-averaged redistribution function. This function describes the elementary act of a photon scattering by a model two-level atom with the upper level broadened due to radiation damping. An eigenvalue and eigenvector determination problem is formulated and the relevant matrices are found analytically. Numerical procedures for their computations are elaborated as well. A simple method for the numerical calculations accuracy evaluation is suggested. It is shown that a family of redistribution functions describing the light scattering process within the spectral line frequencies can be constructed if the eigenvalue problem for the considered function is solved. It becomes possible if the eigenvalues and eigenvectors with the appropriate basic functions are used. The Voigt function and its derivatives used as basic functions are studied in detail as well.

1. Introduction

In the theory of spectral lines formation various assumptions concerning the photon-atom elementary interaction mechanism had been considered. Historically the simplest assumption assuming the elementary act of scattering not affecting the photon energy was considered first. This version of the simplified interaction is referred usually as the coherent or monochromatic scattering. Mathematical methods for solving the light scattering problems had been progressively developed especially for this physical assumption.

However, it is clear, that any physical interaction between systems of physical objects, generally speaking, leads to the energy exchange between interacting objects. Therefore, it was obvious from the very beginning that any realistic consideration of the light scattering issues should take into account the possible energetic exchanges described by the known cross sections of the considered processes. To our knowledge, Eddington was the first to point to the importance of non-coherency when the light scattering in spectral line frequencies is considered [1].

That was requiring a new approach for description of photon's energy change due to its interaction with the scattering medium. The simplest from the mathematical description viewpoint assumption is, so called, complete frequency redistribution of photons, having a rather transparent physical meaning: the scattering atoms “do not remember” the energetic status of absorbed photons and, therefore, photons are re-emitted with portions of energy according to the emission coefficient depending on the atom's internal features only. Because of that not any correlation exists between the frequencies of absorbed and re-emitted photons. This approximation was suggested in early 40 s of the last century [2].

In general, as follows from the quantum electrodynamics principles, any single interaction between a photon and a scattering quantum-mechanical object depends on the geometry of interaction and the energies of interacting particles. Hence, one should take all the relevant factors into account. For the further simplification

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purposes a new probabilistic quantity instead of physical cross sections was introduced into the radiative transfer theory, called redistribution function. Moreover, mostly the angle-averaged redistribution function denoted $r(x',x)$ is used in practice which considers the medium physical conditions only implicitly. As far as we know, the first classification of derived in that way redistribution functions has been done by Hummer [3] who has considered different physical cases separately.

It is noteworthy, that the direct calculations of the redistribution functions sometimes are rather laborious and their application for practical usage encounters huge difficulties. Therefore, mostly some approximate methods are carried out for their calculations providing required qualitative and quantitative accuracy for comparatively less efforts. The method of redistribution functions substitution for their bilinear expansion has the advantage that allows one to test different approaches of the truncated sums instead of the corresponding infinite series. On the other hand, the bilinear expansions of redistribution functions are extremely handy tools if one uses the Principle of Invariance (PI) to solve the radiative transfer problems (see, for example, [4–8] and references therein). Therefore, we consider the possibility of building an "artificial" expansion, in this paper, for a redistribution function never expanded into bilinear series.

2. The redistribution function $r_{II}(x',x)$

Let us first redefine the redistribution function $r(x',x)$ which has a rather simple physical meaning: the quantity $r(x',x)dx$ represents the probability that a photon with the dimensionless frequency x' will be absorbed by an atom and re-emitted then in the frequency interval $(x;x+dx)$. The introduced dimensionless frequencies show the distance of photon's frequency $\nu(\nu')$ from the line center frequency ν_0 in Doppler half widths $\left(x=\frac{\nu-\nu_0}{\Delta\nu_D}\right)$. This redistribution function differs from one defined by Hummer [3] by the constant factor $\left(\pi^{\frac{1}{4}}U(0,\sigma)\right)^{-1}$, where the function

$$U(x,\sigma)=\frac{\sigma}{\pi}\int_{-\infty}^{\infty}\frac{\exp(-t^2)}{(x-t)^2+\sigma^2}dt, \tag{1}$$

is the well known Voigt function and $\sigma=\frac{\Delta\nu_T}{\Delta\nu_D}$, where $\Delta\nu_T$ is the total half-width of the line caused by all the broadening mechanisms taken into account.

The redistribution function describing the photon scattering within the line frequencies of the model two-level atom the upper level of which is broadened due to radiation damping has been independently derived by Henyey [9], Unno [10] and Sobolev [11] assuming that in the atom's reference frame the scattering is coherent. Then, using also Hummer's [3] designation one can represent it in the following form:

$$r_{II}(x',x)=\frac{1}{\pi U(0,\sigma)}\int_{\frac{|\overline{x}-\underline{x}|}{2}}^{\infty}\exp(-t^2)\left[\arctan\frac{\underline{x}+t}{\sigma}-\arctan\frac{\overline{x}-t}{\sigma}\right]dt. \tag{2}$$

In the expression (2) we used the following notations: $\overline{x}=\sup(x',x)$ and $\underline{x}=\inf(x',x)$.

It is noteworthy that there have been known bilinear expansions for two out of four redistribution functions described in the Hummer [3], namely, $r_I(x',x)$ and $r_{III}(x',x)$ before the paper [3] appeared. However, up to nowadays not any "natural" bilinear expansion has been revealed for the function $r_{II}(x',x)$. Therefore, we are trying to create such a bilinear expansion on the base of an artificial procedure using for that a system of some appropriate basic orthogonal functions for building of corresponding eigenfunctions.

In order to construct numerically such an expansion, let us first introduce here another representation of $r_{II}(x',x)$ derived by Nikoghossian [12] (see, also, Heinzel [13])

$$r_{II}(x',x)=\frac{\sigma}{\pi U(0,\sigma)}\int_{-\infty}^{\infty}\frac{r_I(x'+t,x+t)}{t^2+\sigma^2}dt. \tag{3}$$

Taking into account the following expression for the δ-function:

$$\lim_{\sigma\to 0}\frac{\sigma}{\pi}\frac{1}{t^2+\sigma^2}=\delta(t), \tag{4}$$

one finds easily that the function $r_{II}(x',x)$ transforms into the $r_I(x',x)$ when $\sigma=0$.

On the other hand, the function $r_I(x',x)$ allows the following bilinear expansion first derived by Unno [14] (see also [3]):

$$r_I(x',x)=\int_{|\overline{x}|}^{\infty}\exp(-t^2)dt=\sum_{k=0}^{\infty}\frac{\alpha_{2k}(x')\alpha_{2k}(x)}{2k+1}, \tag{5}$$

where

$$\alpha_k(x)=(2^k\pi^{\frac{1}{2}}k!)^{-\frac{1}{2}}H_k(x)\exp(-x^2) \tag{6}$$

and $H_k(x)$ are the Hermit polynomials.

The obvious connection between functions $r_{II}(x',x)$ and $r_I(x',x)$ expressed by relation (3) allows to suggest the functions (6) as basic ones for constructing the eigenfunctions of $r_{II}(x',x)$. Taking into account this connection, one can search the bilinear expansion for $r_{II}(x',x)$ in the following form:

$$r_{II}(x',x)=\sum_{k=0}^{\infty}\frac{\omega_{2k}(x',\sigma)\omega_{2k}(x,\sigma)}{\zeta_k(\sigma)}, \tag{7}$$

where

$$\omega_{2k}(x,\sigma)=\sum_{m=0}^{\infty}\gamma_{km}(\sigma)\alpha_{2k}(x). \tag{8}$$

The vector $\zeta_k(\sigma)$ and matrix $[\gamma_{km}(\sigma)]$ are, correspondingly, the eigenvalues and eigenfunctions of the following problem (see, for example, [15,4]):

$$\sum_{m=0}^{\infty}\left[\gamma_{km}(a_{mn}-\zeta_k(\sigma)b_{mn})\right]=0, \tag{9}$$

where

$$a_{mn}=\int_{-\infty}^{\infty}\alpha_{2m}(x)\alpha_{2n}(x)dx, \tag{10}$$