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Application of general invariance relations reduction method to solution of radiation transfer problems

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ABSTRACT

A brief analysis of different properties and principles of invariance to solve a number of classical problems of the radiation transport theory is presented. The main ideas, constructions, and assertions used in the general invariance relations reduction method are described in outline. The most important distinctive features of this general method of solving a wide enough range of problems of the radiation transport theory and mathematical physics are listed. To illustrate the potential of this method, a number of problems of the scalar radiative transfer theory have been solved rigorously in the article. The main stages of rigorous derivations of asymptotical formulas for the smallest in modulo elements of the discrete spectrum and the eigenfunctions, corresponding to them, of the characteristic equation for the case of an arbitrary phase function and almost conservative scattering are described. Formulas of the same type for the azimuthal averaged reflection function, the plane and spherical albedos have been obtained rigorously. New analytical representations for the reflection function, the plane and spherical albedos have been obtained, and effective algorithms for calculating these values have been offered for the case of a practically arbitrary phase function satisfying the Hölder condition. New analytical representation of the «surface» Green function of the scalar radiative transfer equation for a semi-infinite plane-parallel conservatively scattering medium has been found. The deep regime asymptotics of the "volume" Green function has been obtained for the case of a turbid medium of cylindrical form.

1. Introduction

The properties of symmetry and invariance have been widely used in explicit and implicit forms practically in all fields of people activity for many centuries and even many thousands of years [1]. However, they play the most significant role in physics, mathematics and mathematical physics (MP). Moreover, these properties and different principles of the invariance form the basis of the essential part of fundamental theoretical constructions used in these sciences. It should be noted that in the main by properties and principles of invariance are meant general statements on invariance of some objects, systems, constructions, physical laws, solutions, and so on with respect to sets of some actions or operations. Further by invariance relations (or general invariance relations) are meant relations that are consequences of properties (or principles) of invariance and connect solutions of different problems or problems of the same type. Usually, in the natural sciences researchers take (in an explicit or implicit form) groups (in the algebraic sense) as a set of actions or operations used to formulate the principles of invariance. For example, Galilei's and Einstein–Poincare's relativity principles are the principles of invariance in which the invariance of the

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forms of physical laws (with respect to transforms of Galilei's group and the proper Poincare's group) is stated. However, not all of the properties and principles of invariance can be formulated within the framework of the group-theoretical approach (see, for example, [1–7] and references therein). One of the reasons for such situation is the fact that a knowledge of only physical laws for unambiguous (in dynamic or statistical sense) description of real processes or phenomena is insufficient [1,7] as it is necessary to specify initial and (or) boundary conditions and some other conditions (for example, such as the classes of functions in which the solutions of problems are found). On the other hand, there exists an essential general fundamental constructive property Υ of the universe (and its parts) and methods of its knowledge. The meaning of the property Υ was explained in detail in [1,7]. This property can mean the existence of some nonempty set W (its elements are actions or operations) for any nonempty set A that may make sense of a set of objects (for example, set of turbid media), processes, laws, images, designs, solutions, systems, etc. It should be emphasized that all elements of the set W make sense of real or mental actions or operations of two types. The actions or operations of the first type dismember or decompose of the set A into its components. The second type includes actions or operations that produce association, make a synthesis or imbedding of any selected parts of the set A into an organic whole (it can contain the set A and its subsets). We specially emphasize that the sets constructed from the set A (or its subsets) with the use of the above-mentioned actions or operations from the set W should not change some properties of the subsets of the set A or the set A itself (i.e., some of these properties are invariant with respect to actions or operations of the set W). From the algebraic point of view, the set W cannot always be interpreted as a group. Sometimes the set W cannot be imbedded into any group [1,7]. The property Υ is a necessary (but not sufficient) condition for the existence of representations about symmetry and this one is more general than the properties of symmetry [1,7]. It should be noted that the invariant (or partially invariant) dismemberment and decomposition of the set A and its parts can sometimes be made by using group or semigroup operations or equivalence relations [1,7]. In fact, the above-mentioned properties (in particular, the property Υ), principles of invariance and sets are widely used in explicit or implicit forms when solving a number of very complicated problems of mathematics, physics, and MP. The property Υ was often considered as a nearly self-evident property.

It is obvious that the scientific value of any property or principle of invariance depends on the nontriviality of the results obtained on their basis. However, to obtain such results it is necessary previously to carry out effective procedures of finding consequences of such properties and principles. Generally speaking, the development of the procedures of this kind is not always a simple task. In virtue of the afore-said, efficient procedures for obtaining these consequences are an important part of any method that uses the properties and principles of this kind. Among the common and useful features of methods that use the properties and principles of invariance we can distinguish their heuristic nature and generality. In particular, solutions of a number of difficult problems of mathematics, physics and MP were obtained due to the heuristic nature of these methods (see, for example, [1,7]). All the foregoing is true within the framework of the optics and the radiation transport theory (RTT).

Further, to additionally elucidate and substantiate the above-mentioned assertions, we consider a number of important works in which some simple versions of the actions (or operations), properties (or principles) of invariance and of effective procedures were used. Some of constructions and ideas of these works were partially taken into account in developing and using the general invariance relations reduction method (see, for example, Refs. [5–19] and references therein). This method (GIRRM [7]) allows one to effectively solve a number of multidimensional boundary-value problems [1,7,10–14,18,19] for the radiative transfer equation (RTE) and some problems [15–18] of MP.

We begin with a brief description of a number of results obtained by Stokes [20]. He solved the following problem: “There are m parallel plates each reflects and transmits given fractions r, t of the light incident upon it (light of the unity intensity being incident on the system); it is required to find the intensities of the reflected and refracted light”. Stokes implicitly used, in fact, operations of invariance partition of the system of $(m+n)$ plates into two subsystems of m and n plates and, considering their “interaction”, obtained a system of functional equations satisfied by the required quantities. Unlike his predecessors, Stokes comprehensively researched the problem and took into account the possible presence of light absorption within a plate (but in the article [20] it was assumed that the scattering of light within the plate was lacking). In fact, an invariant, which is independent of the choice of natural numbers m and n, was found in [20] in an explicit form. Stokes obtained this invariant using the property of symmetry of the left-hand part of Eq. (6) in [20] with respect to the permutation of m and n. A detailed analysis of the results of [20] and earlier publications by other authors was performed by Tuckerman [21]. In [21], Stokes's formulas were derived by some other way and the relationships between the results of Stokes, elements of the theory of difference equations and the simplest version of the classical method of invariant imbedding subsequently proposed by Bellman and Kalaba [2] were noted.

Significant constructive steps towards an effective use of the properties of invariance to solve the problems of the scalar radiative transfer theory (SRTT) for the case of a macroscopically homogeneous plane-parallel turbid medium were made by Ambartsumian [22–27] and Chandrasekhar [28]. In [22,23], in deriving nonlinear integral equations (for the cases of isotropic and anisotropic scattering) for a reflection function (or functions, through which it is expressed) corresponding to a semi-infinite plane-parallel medium $V_{[0,+\infty)}$, one almost obvious property of the invariance of this function was first used in an explicit form. The essence of this property is that the reflection function does not change if an additional layer, which has the same local optical characteristics as the