Journal of Quantitative Spectroscopy & Radiative Transfer 178 (2016) 269–275

Contents lists available at ScienceDirect

Journal of Quantitative Spectroscopy & Radiative Transfer

journal homepage: www.elsevier.com/locate/jqsrt

Numerical comparison of spectral properties of volume-integral-equation formulations

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ARTICLE INFO

Article history:
Received 15 September 2015
Received in revised form
14 January 2016
Accepted 14 January 2016
Available online 25 January 2016

Keywords:
Spectrum of integral operator
Numerical modeling
Volume-integral-equation

ABSTRACT

We study and compare spectral properties of various volume-integral-equation formulations. The equations are written for the electric flux, current, field, and potentials, and discretized with basis functions spanning the appropriate function spaces. Each formulation leads to eigenvalue distributions of different kind due to the effects of discretization procedure, namely, the choice of basis and testing functions. The discrete spectrum of the potential formulation reproduces the theoretically predicted spectrum almost exactly while the spectra of other formulations deviate from the ideal one. It is shown that the potential formulation has the spectral properties desired from the preconditioning perspective.

1. Introduction

Electromagnetic scattering problems involving inhomogeneous objects are often solved by the volume-integral-equation methods (VIE). The VIE method is suitable for complicated scattering problems due to its simplicity; only Green's function of the background is required. Moreover, the radiation condition is automatically satisfied. The drawback in the VIEs is that the discretization procedure leads to a full matrix equation in contrast to, for example, the finite-element method where the system is sparse. This implies that to obtain a solution for the matrix equation is of order $\mathcal{O}(N^3)$ complexity for time and $\mathcal{O}(N^2)$ for memory, where N is the number of unknowns. For the lowest order basis, typically 10 unknowns per wavelength are needed.

High computational complexity prevents the usage of the direct VIE solvers for large structures. Using an iterative method such as the conjugate-gradient (CG) or generalized minimal residual (GMRES) method, the solution time is reduced to $\mathcal{O}(MN^2)$ where M is the number of iterations needed to solve the system. By accelerating the matrix–vector multiplication required in each iteration step with, e.g., a fast multilevel multipole algorithm (MLFMA) or fast Fourier transform (FFT) based techniques [1–5], the computational complexity is reduced to $\mathcal{O}(MN\log N)$ for time and $\mathcal{O}(N)-\mathcal{O}(N\log N)$ for memory. Hence, solution time becomes manageable if $M \ll N$.

For an efficient algorithm it is necessary that the number of iterations M is much smaller than N. M depends on the conditioning of the matrix which, in turn, depends on materials, size, and shape of the scatterer. Unfortunately, the number of iterations increases rapidly with respect to the permittivity and size [6–9]. To understand reasons for this, we need to study the spectrum of the integral operator. Theoretically, the spectrum of the volume integral operator has been studied in [10–12]. These studies show that the spectrum and the spectral radius depend on the permittivity function. The spectral radius, in turn, defines the conditioning of the matrix, and consequently, the convergence of the iterative solution.

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http://dx.doi.org/10.1016/j.jqsrt.2016.01.020
0022-4073/

The spectral properties of the discrete system (the eigenvalues of the matrix) depend on the discretization technique employed, hence, it is important to study the effects of discretization numerically. Numerical studies, however, have been restricted to a couple of the most popular volume-integral-equation formulations and discretizations. The discrete-dipole-approximation (DDA) type formulations with cubic elements were analyzed in [10,13,14], and with rectangular elements in [15]. Moreover, the spectral properties of the electric current J-VIE with the L^2 Galerkin dicretization for tetrahedral mesh have been studied [16]. It is worth noting that the integration of Green's tensor (IGT) formulation of the DDA is almost equivalent to the J-VIE discretized with cubic elements and point matching. The only difference is that the DDA maps the polarization current to the electric field and the J-VIE maps the current to itself, hence, the matrix elements differ by a factor of (ϵ_r-1).

In this paper, we compare the eigenvalue distributions computed by four VIE formulations and their standard discretizations. We consider the electric flux density formulation (D-VIE) discretized with the SWG (Schaubert–Wilton–Glisson) basis and testing functions [17], field formulation (E-VIE) with the curl conforming basis and testing functions [18,19], current formulation (J-VIE) [20] with L^2 basis and testing functions, and potential formulation (P-VIE) with scalar and vector H^1 basis function and point matching [21]. We apply linear tetrahedral elements for discretizations and the results presented here cannot be directly generalized to other element shapes such as rectangular or curvilinear elements.

2. Formulations

Consider time-harmonic electromagnetic wave scattering by an inhomogeneous dielectric object bounded by volume V in free space. The time factor of $\exp(-i\omega t)$ is assumed and suppressed. The relative permittivity $\epsilon_r(\boldsymbol{r})$ may be a function of position in V. The background is homogeneous with constant ϵ_0 and μ_0. Let us define the volume potential operator as

$$\mathcal{V}(\boldsymbol{F})(\boldsymbol{r}) = \int_V G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{F}(\boldsymbol{r}')\,dV', \tag{1}$$

where G is Green's function of the background. By using the volume-equivalence principle, the following representations are obtained for the total electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ fields [22]:

$$\begin{aligned}\boldsymbol{E} &= \boldsymbol{E}^{inc} + \frac{-1}{i\omega\epsilon_0}\left(\nabla\nabla + k^2\overline{\overline{I}}\right)\cdot\mathcal{V}(\boldsymbol{J}) - \nabla\times\mathcal{V}(\boldsymbol{M})\\ \boldsymbol{H} &= \boldsymbol{H}^{inc} + \frac{-1}{i\omega\mu_0}\left(\nabla\nabla + k^2\overline{\overline{I}}\right)\cdot\mathcal{V}(\boldsymbol{M}) + \nabla\times\mathcal{V}(\boldsymbol{J})\end{aligned} \tag{2}$$

in which $\boldsymbol{E}^{\text{inc}}$ and $\boldsymbol{H}^{\text{inc}}$ denote the incident fields with sources outside the object. The source functions in (2) are the equivalent electric and magnetic current densities:

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{r}) &= -i\omega\epsilon_0(\epsilon_r(\boldsymbol{r})-1)\boldsymbol{E}(\boldsymbol{r})\\ \boldsymbol{M}(\boldsymbol{r}) &= -i\omega\mu_0(\mu_r(\boldsymbol{r})-1)\boldsymbol{H}(\boldsymbol{r}).\end{aligned} \tag{3}$$

From now on we assume that the permeability $\mu_r = 1$, hence the magnetic current $\boldsymbol{M}$ is identically zero. Based on the representations in (2), we can derive three VIE formulations. The most widely used formulation is the D-formulation or D-VIE in which the unknown function is the flux density $\boldsymbol{D}$ [17,23]. By representing the equivalent current $\boldsymbol{J}$ in terms of the flux density and inserting it into (2), the D-VIE is obtained:

$$\epsilon_0\boldsymbol{E}^{\text{inc}} = \overline{\overline{\epsilon}}_r^{-1}\cdot\boldsymbol{D} - (\nabla\nabla + k^2\overline{\overline{I}})\cdot\mathcal{V}(\overline{\overline{\chi}}\cdot\boldsymbol{D}). \tag{4}$$

Here the material parameter $\overline{\overline{\chi}} = \overline{\overline{I}} - \overline{\overline{\epsilon}}_r^{-1}$.

The integral-equation can be written for the equivalent polarization current $\boldsymbol{J}$ which is the actual source for the scattered fields [20]. We call this formulation as the J-formulation or J-VIE, and it reads as

$$\boldsymbol{J}^{\text{inc}} = \boldsymbol{J} - \overline{\overline{\tau}}\cdot(\nabla\nabla + k^2\overline{\overline{I}})\cdot\mathcal{V}(\boldsymbol{J}), \tag{5}$$

where $\overline{\overline{\tau}} = \overline{\overline{\epsilon}}_r - \overline{\overline{I}}$.

To derive the electric field E-formulation (E-VIE), we use the identity

$$(\nabla\nabla + k^2\overline{\overline{I}})\cdot\mathcal{V}(\boldsymbol{F}) = \nabla\times(\nabla\times\mathcal{V}(\boldsymbol{F})) - \boldsymbol{F}, \tag{6}$$

since it is more natural to apply the curl rather than div-operator to the electric field. Representing the unknown in terms of the electric field, the E-VIE can be written as follows: [19,18]:

$$\boldsymbol{E}^{\text{inc}} = \overline{\overline{\epsilon}}_r\cdot\boldsymbol{E} - \nabla\times\nabla\times\mathcal{V}(\overline{\overline{\tau}}\cdot\boldsymbol{E}). \tag{7}$$

Finally, the integral-equation for the vector $\boldsymbol{A}$ and scalar ϕ potentials, defined as

$$\boldsymbol{E} = i\omega\boldsymbol{A} - \nabla\phi, \quad \boldsymbol{H} = \mu_0^{-1}\nabla\times\boldsymbol{A}, \tag{8}$$

can be derived by applying the Lorentz gauge

$$i\omega\nabla\cdot\boldsymbol{A} = -k^2\phi \tag{9}$$

in which case both potentials satisfy the Helmholtz equation. Using the volume-equivalence principle, the potential formulation P-VIE can be written as [24]

$$\begin{cases} i\omega\boldsymbol{A}^{inc} &= i\omega\boldsymbol{A} - k^2\mathcal{V}[\overline{\overline{\tau}}\cdot(i\omega\boldsymbol{A} - \nabla\phi)]\\ \phi^{inc} &= \phi - \mathcal{S}[\overline{\overline{\tau}}\cdot(i\omega\boldsymbol{A} - \nabla\phi)]\end{cases}. \tag{10}$$

in which

$$\mathcal{S}(\boldsymbol{F})(\boldsymbol{r}) = \int_S \boldsymbol{n}(\boldsymbol{r}')\cdot\boldsymbol{F}(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')\,dS', \tag{11}$$

and $\boldsymbol{n}'$ is the outer unit normal vector of the surfaces S on which the permittivity is discontinuous.

3. Discretizations

In this section, we consider discretizations of the D-, J-, E-, and P-formulations. Let us divide the object with linear tetrahedral elements, and define the basis $\boldsymbol{b}$ and the testing $\boldsymbol{t}$ functions on the tetrahedral mesh. The residual error is forced to be orthogonal to test functions with the symmetric L^2 product:

$$\langle\boldsymbol{F},\boldsymbol{G}\rangle = \int_V \boldsymbol{F}\cdot\boldsymbol{G}\,dV, \tag{12}$$

where V is the volume of the object.