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Solution of the 2-D steady-state radiative transfer equation in participating media with specular reflections using SUPG and DG finite elements

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ABSTRACT

The 2D radiative transfer equation coupled with specular reflection boundary conditions is solved using finite element schemes. Both Discontinuous Galerkin and Streamline-Upwind Petrov–Galerkin variational formulations are fully developed. These two schemes are validated step-by-step for all involved operators (transport, scattering, reflection) using analytical formulations. Numerical comparisons of the two schemes, in terms of convergence rate, reveal that the quadratic SUPG scheme proves efficient for solving such problems. This comparison constitutes the main issue of the paper. Moreover, the solution process is accelerated using block SOR-type iterative methods, for which the determination of the optimal parameter is found in a very cheap way.

1. Introduction

Thermal radiation is a heat transfer mode important to take into account in many practical high temperature engineering applications such as, in modelling industrial furnaces, combustion chambers, or forming processes, to cite but a few [1,2]. The forward model commonly used to model the propagation of thermal radiation within semi-transparent participating media is the so-called radiative transfer equation (RTE). This equation is integro-differential, so that its solution is far from being given straightforwardly, especially when the geometry of the bounded domain cannot be considered as mono-dimensional. In such cases, the use of numerical and physical approximation methods is mandatory to access the solution of the RTE.

In the field of numerical methods, for the solution of radiative transfer problems in participating media, the finite volume method (FVM) for the space discretization, coupled with the discrete ordinated method (DOM) or other methods, are among the most widely used. Indeed, they can provide good accuracy in a wide range of practical problems with moderate computational requirements [3–6]. The review paper [7] lists recent advances in such numerical methods for the solution of the RTE with FVM.

Other methods that have been used to solve the RTE include, the zonal method [8], natural element and meshless methods [9,10], or the finite difference method [11].

Besides above cited methods, the finite element method (FEM) has a growing attention mainly because it is based on the variational formulation, thus allowing theoretical studies such as existence, uniqueness and stability of the solution. Also, very complex geometries can be dealt with, and FEM can be versatile as soon as the variational formulation is written down in a general framework. For instance, using appropriate finite element libraries such as

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Nomenclature

β	extinction coefficient
κ	absorption coefficient
Φ	scattering phase function
ρ	reflectivity coefficient
σ_s	scattering coefficient
$\tilde{n}_i$	complex refraction index
$\tilde{L}$	incoming radiance
G	fluence
L	radiance intensity
L_b	emitting radiance of blackbody

Angle and space

$\boldsymbol{n}$	outward unit normal vector of a boundary
$\boldsymbol{s}$	direction
ω_i	quadrature weight
$\boldsymbol{x}$	space coordinates
δ, γ	proportion of solid angle
$\Delta\theta$	angle between two nearby discrete directions
$\mathcal{C}_i$	cell i
$\mathcal{D}$	space of medium
Ω	solid angle
$\partial\mathcal{D}_*$	part of boundary of medium
$\mathcal{S}^{n-1}$	unit circle
h	maximal size of triangles of a mesh
N_d	number of discrete directions
N_e	number of mesh elements
r	relaxation parameter
v	test function

Subscripts and superscripts

$+$	boundary for outgoing radiance
$-$	boundary for incoming radiance
inc	incident direction
in	input radiance
out	output radiance
ς	permutation function
m, j, k	discrete direction of radiation
N	number of the iterations

those in [12], for a given variational formulation, the theoretical development of the code with quadratic Lagrange functions is not more expensive than with linear Lagrange functions. In the same spirit, the change of boundary condition location or sources can be performed in a very straightforward way, as well as the modification of physical properties and so on, because, basically, one of the main effort resides in writing down the variational formulation in a general framework. This has been used many times in the last decade for the RTE, for instance in [13–15].

The RTE being an equation in which the advection operator plays a central role, it might sometimes be inappropriate to employ ordinary FEM due to the presence of oscillations, especially when the albedo gets high. In order to cope up with such a difficulty, decentered schemes such as the Least-Squares (LS) or Streamline Upwind Petrov–Galerkin (SUPG) allows to drastically avoid oscillations, but at the price of adding artificial numerical diffusion. Recently, the LS-FEM and related SUPG schemes have been used in [16–21] mostly in view of optical tomography applications. Note that in most of these papers the free output boundary condition was considered, not the one allowing reflections. Also, another difficulty arising with pure LS-FEM is that the number of terms in the variational formulation can become very high, especially when considering the specular boundary condition as it is the case in this study. That is the reason why the SUPG scheme, which is a purely decentered finite element scheme is the subject of this study.

Another possibility for the solution of the RTE based on variational formulations is to use a Discontinuous Galerkin (DG) formulation. The DG method, firstly developed in the field of neutron transport [22], has then been used for solving the transport problem of radiation [23–28]. This method is very attractive because it has all the advantages of the FEM and, moreover, it is also elementwise conservative, such as the FVM.

Based on the above-cited methods, several extensions have been developed for instance the Multi-Scale Finite Element Method [29] with mesh adaptation or the use of vector radiative transfer equation to adjust atmosphere and surface properties [30].

As far as we are concerned, the numerical developments are performed in view of radiative characterization of multi-dimensional materials such as open-cell foams for instance [31]. Such characterization relies on solving an inverse problem that demands numerous iterations of "forward" model while changing the physical properties. As a consequence, the numerical tools must lead to a simulation that is (i) accurate in the sense that the approximated numerical solution must be close enough to the solution of the continuous problem, (ii) robust in the sense that the solution is of equal quality in various situations (physical properties, boundary conditions, etc.), and (iii) effective in the sense that the solution is robust and accurate in a reasonable CPU time.

The physical model of concern is now precised. At a given temperature, the RTE problem consists in searching the radiance $L(\boldsymbol{x},\boldsymbol{s})$ in a medium $\mathcal{D}$ such that:

$$(\boldsymbol{s}\cdot\nabla+\kappa+\sigma_s)L(\boldsymbol{x},\boldsymbol{s})-\sigma_s\oint_{\mathcal{S}^{n-1}}L(\boldsymbol{x},\boldsymbol{s}')\Phi(\boldsymbol{s},\boldsymbol{s}')\,\mathrm{d}\boldsymbol{s}'=\kappa L_b \quad \forall\boldsymbol{s}\in\mathcal{S}^{n-1} \tag{1}$$

where $\boldsymbol{s}\in\mathcal{S}^{n-1}$ is the direction of propagation of L at the location $\boldsymbol{x}$ ($\mathcal{S}^{n-1}$ is the unit circle in 2D), κ and σ_s are the homogeneized absorption and scattering coefficients, respectively, and $\Phi(\boldsymbol{s},\boldsymbol{s}')$ is the scattering phase function. The spectral dependance of physical properties (and thus of the radiance) is omitted for clarity considerations.

In applications considered here, the medium is illuminated with a collimated beam (i.e. with $\boldsymbol{s}=\boldsymbol{s}_{\text{in}}$ and $\boldsymbol{s}\cdot\boldsymbol{n}<0$