Journal of Quantitative Spectroscopy & Radiative Transfer 166 (2015) 81–92

Contents lists available at ScienceDirect

Journal of Quantitative Spectroscopy & Radiative Transfer

journal homepage: www.elsevier.com/locate/jqsrt

Generalization of the optical theorem for monochromatic electromagnetic beams of arbitrary wavefront in cylindrical coordinates

F.G. Mitri

Chevron, Area 52 Technology-ETC, Santa Fe, NM 87508, USA

ARTICLE INFO

Article history:
Received 19 June 2015
Received in revised form
22 July 2015
Accepted 24 July 2015
Available online 3 August 2015

Keywords:
Optical theorem
Extinction
Cross-section
Efficiency
Scattering
Absorption

ABSTRACT

The optical theorem constitutes of the fundamental theorems in optical, acoustical, quantum, and gravitational wave scattering, which relates the extinction cross-section to the forward scattering complex amplitude function of plane waves. In this analysis, a generalized formalism is presented for beams of arbitrary character in cylindrical coordinates without restriction to the plane wave case of the angles of incidence and scattering. Based on the partial-wave series expansion method of cylindrical multipole, analytical expressions for the extinction, absorption, scattering cross-sections and efficiency factors are derived for an object of arbitrary shape. An "interference scattering" term arises in the cross-section (or efficiency), which describes the mutual interference between the diffracted or specularly reflected waves. Examples for plane waves and 2D scalar quasi-Gaussian focused beams are also considered, which illustrate the theory. The generalized optical theorem in cylindrical coordinates can be applied to evaluate the extinction efficiency from any object of arbitrary geometry placed on or off the axis of the incident beam. Applications in the context of wave scattering theory by a single particle or multiple particles would benefit from the results of the present study, in addition to other phenomena such as the radiation force and torque.

1. Introduction

From the standpoint of wave scattering theory, the statement of conservation of energy applied to the extinction phenomenon, accounting for both absorption and scattering by a particle in any form of wave propagation, is written as [1, p. 13] $\sigma_{ext} = \sigma_{abs} + \sigma_{sca}$, where specific cross-sections have been defined with each of these processes. Understanding these phenomena during wave transport allows quantitative characterization of the media of wave propagation by deconvolution of scattering and attenuation spectra.

The "optical theorem" (OT) [1–8], which is otherwise known as the "extinction theorem" [1], relates the extinction cross section of an object of arbitrary geometry placed in the field of monochromatic plane waves to its forward scattering amplitude, which is the scattered wave amplitude measured in the far field along the forward direction of wave propagation [9, p. 20]. The extension of the optical theorem to account for the plane waves scattered at *any* angle (i.e., not only restricted to the forward or backwards directions) has been obtained throughout an integral relation for the angle-dependent scattering amplitude from the standpoint of quantum theory [10–12], which constitutes a "generalized optical theorem" (GOT). Subsequently, it has been applied in the context of electron diffraction theory [13], scalar optical [14] and evanescent waves [15]. Additional extensions for the GOT to study vectorial wave phenomena related to electromagnetic waves [16,17], elastic waves [18], surface waves [19] and layered media [20] have been formulated as well.

E-mail address: F.G.Mitri@ieee.org

http://dx.doi.org/10.1016/j.jqsrt.2015.07.016
0022-4073/

The relevance of the GOT relies in the quantitative and accurate evaluation of cross sections that account for the redistribution of the energy, rather than numerical integration procedures [8]. Essentially, the extinction, scattering, and absorption cross sections (or power) are meaningful measures of the object scattering and absorption properties. As such, applications and experimental methods including (but not limited to) medical, non-destructive, sonar and underwater imaging, multiphase flow characterization, spectroscopy and particle sizing in dilute suspensions, near-field diffraction x-ray and ultrasonic tomography, microscopy, and inverse scattering, to name a few, would benefit from an improved GOT formalism that accounts for the character of the incident wavefront, since all the applications involve the use of devices emitting tailored wavefronts as opposed to plane unbounded waves.

Several formulations exist for acoustical [21,22], quantum [23], and optical Gaussian beams [24] (and others of arbitrary shape [25]) that possess some degree of amplitude roll-off in the transverse direction. Though in principle those formalisms are applicable to any object of arbitrary shape, computing the extinction, scattering, and absorption cross sections (or their corresponding efficiencies) of elongated (cylindrical-like) objects with the analytical models in spherical coordinates, leads to numerical instabilities and significant inaccuracies. Therefore, it is of some importance to develop a generalization procedure of the optical theorem to produce an analytical formalism suitable for elongated objects in cylindrical coordinates.

The aim of this analysis is therefore directed toward this goal, in order to provide a generalized formulation in cylindrical coordinates applicable to any electromagnetic beam of arbitrary character, interacting with a scatterer of arbitrary geometry and size, located arbitrarily in space.

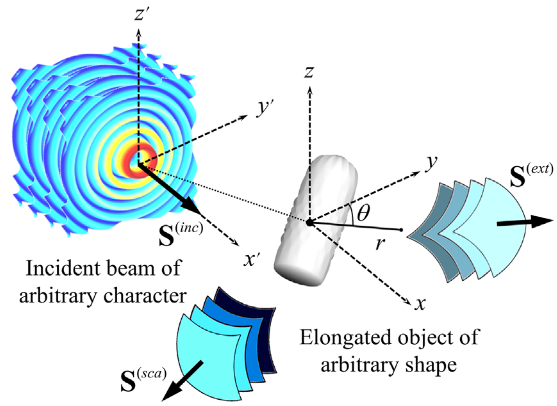

Fig. 1. An elongated object of arbitrary shape placed in the field of an incident electromagnetic beam of arbitrary wavefront in cylindrical coordinates (r, θ, z). The primed coordinate system has its origin at the center of the beam, while the unprimed coordinate system is referenced to the object. The Poynting vectors $\mathbf{S}^{(sca)}$ and $\mathbf{S}^{(ext)}$, associated with the scattered and interacting waves, respectively, are used to derive the expressions for the specific cross-sections (or efficiencies).

2. Theoretical analysis

Consider an unpolarized electromagnetic vectorial beam of angular frequency ω incident along an arbitrary direction on a viscoelastic object of arbitrary geometry immersed in a lossless medium (Fig. 1). A time-dependence in the form of $e^{-i\omega t}$ is assumed, but omitted from the equations for convenience.

For an arbitrary-shaped beam composed of monochromatic waves, the incident electric $\mathbf{E}^{(i)}$ and magnetic $\mathbf{H}^{(i)}$ vector fields in the frequency domain can be expressed in a cylindrical coordinates system (r, θ, z), respectively, as [26,27]

$$\mathbf{E}^{(inc)}(r,\theta,z) = E_0 \sum_{n=-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \left[A_n(k_z)\mathbf{N}_n^{(inc)}(k_z,r) + B_n(k_z)\mathbf{M}_n^{(inc)}(k_z,r) \right] e^{ik_z z} dk_z \right\} e^{in\theta}, \quad (1)$$

$$\mathbf{H}^{(inc)}(r,\theta,z) = -i\sqrt{\varepsilon_{ext}}E_0 \sum_{n=-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \left[A_n(k_z)\mathbf{M}_n^{(inc)}(k_z,r) + B_n(k_z)\mathbf{N}_n^{(inc)}(k_z,r) \right] e^{ik_z z} dk_z \right\} e^{in\theta}, \quad (2)$$

where E_0 is a measure of the peak electric-field strength (i.e. amplitude) of the beam, $\varepsilon_{ext} \neq 0$ is the dielectric constant of the medium, k_r and k_z are the radial and axial wave-numbers, respectively, defined as $k^2 = k_r^2 + k_z^2$ [27], where k is the wavenumber of the incident radiation, $\mathbf{M}_n^{(inc)}(k_z, r)$ and $\mathbf{N}_n^{(inc)}(k_z, r)$ are independent solenoidal vector solutions of the vector Helmholtz equation [26,27], that will be defined subsequently by Eqs. (6) and (7). The parameters r and θ are the radial distance and polar angle in the (x,y) plane, respectively. In the generalized case of an object of arbitrary shape in three-dimensions, the radial distance $r = r(\theta, z)$.

The parameters $A_n(k_z)$ and $B_n(k_z)$ are the non-dimensional beam-shape coefficients (BSCs) that solely describe the incident beam of arbitrary wavefront. They are determined from the axial (i.e. along the z-direction) components of the electric and magnetic fields, respectively, as follows. First, the relationship of the two-dimensional Fourier transforms (in θ and z) of the incident field for each of the components is derived, i.e., $\mathcal{F}\left\{E_z^{(inc)}(k_z), H_z^{(inc)}(k_z)\right\} = \int_0^{2\pi}\int_{-\infty}^{+\infty}\left\{E_z^{(inc)}(z), H_z^{(inc)}(z)\right\} e^{-i(n\theta + k_z z)}\, dz\, d\theta$. Then, their inverse functions $\mathcal{F}^{-1}\left\{\mathcal{F}\left\{E_z^{(inc)}(k_z), H_z^{(inc)}(k_z)\right\}\right\}$ are expressed in terms of a Fourier series and a transform as

$$\left\{E_z^{(inc)}(z), H_z^{(inc)}(z)\right\} = \mathcal{F}^{-1}\left\{\mathcal{F}\left\{E_z^{(inc)}(k_z), H_z^{(inc)}(k_z)\right\}\right\} = \frac{1}{(2\pi)^2}\sum_{n=-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}\mathcal{F}\left\{E_z^{(inc)}(k_z), H_z^{(inc)}(k_z)\right\} e^{ik_z z}\, dk_z\right] e^{in\theta},$$

and the results are equated with the expressions for the *axial* electric and magnetic fields (i.e. along the z-direction), respectively, leading to the following expressions for the BSCs as

$$\left\{ \begin{array}{c} A_n(k_z) \\ B_n(k_z) \end{array} \right\} = \frac{k^2}{(2\pi)^2 k_r^2 J_n(k_r r)} \times \int_0^{2\pi}\left[\int_{-\infty}^{+\infty}\left\{ \begin{array}{c} E_z^{(inc)}(r,\theta,z)/E_0 \\ H_z^{(inc)}(r,\theta,z)/H_0 \end{array} \right\} \times e^{-ik_z z}\, dz\right] e^{-in\theta}\, d\theta, \quad (3)$$

where $J_n(\cdot)$ is the cylindrical Bessel function of the first kind, and $H_0 = -iE_0\sqrt{\varepsilon_{ext}}$.