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Two-dimensional axisymmetric formulation of high order spherical harmonics methods for radiative heat transfer

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ABSTRACT

The spherical harmonics (P_N) method is a radiative transfer equation solver, which approximates the radiative intensity as a truncated series of spherical harmonics. For general 3-D configurations, $N(N+1)/2$ intensity coefficients must be solved from a system of coupled second-order elliptic PDEs. In 2-D axisymmetric applications, the number of equations and intensity coefficients reduces to $(N+1)^2/4$ if the geometric relations of the intensity coefficients are taken into account. This paper presents the mathematical details for the transformation and its implementation on the *OpenFOAM* finite volume based CFD software platform. The transformation and implementation are applicable to any arbitrary axisymmetric geometry, but the examples to test the new formulation are based on a wedge grid, which is the most common axisymmetric geometry in CFD simulations, because *OpenFOAM* and most other platforms do not have true axisymmetric solvers. Two example problems for the new axisymmetric P_N formulation are presented, and the results are verified with that of the general 3-D P_N solver, a Photon Monte Carlo solver and exact solutions.

1. Introduction

The study of radiative heat transfer in high-temperature applications with a strongly varying participating medium has become increasingly important in various practical applications like combustion, manufacturing and environmental systems. The Radiative Transfer Equation (RTE) is an integro-differential equation in six independent variables (3 spatial and 2 directional, and wavenumber) [1], which is exceedingly difficult to solve. As a result, approximate solution methods to the RTE, such as the spherical harmonics method (SHM), discrete ordinates method (DOM), the finite volume method (FVM), or the Monte Carlo method are frequently employed to solve radiation problems. Each of these approximate methods has their well-known advantages and drawbacks. The SHM offers an approximate solution to the radiative transfer equation (RTE) by transforming the RTE into a system of elliptic PDEs. This method approximates the radiative intensity as a truncated series of spherical harmonics that decouple the directional and spatial variations of the intensity field. The SHM has been widely applied to particle transport problems [2–4], and some of the derivations for cylindrical geometries have been presented in [5,6].

For axisymmetric problems, physical quantities such as temperature, heat flux, radiative intensity, and chemical species concentrations vary only radially and axially and are, therefore, two-dimensional. As a result, for many of these applications, the transport equations are solved on a 2-D or a thin wedge 3-D computational domain in order to reduce the computational effort. Like the development of the general P_N method [7,8], the application of axisymmetric formulations of P_N method were limited [9,10]

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because of the cumbersome mathematics. Recently, the general P_N (up to P_7) equations and boundary conditions for 3-D geometries have been formulated [11–13] and solved [14] for various cases including a full cylinder with variable radiative properties and a real flame.

In this paper, the 2-D axisymmetric version of P_N and its boundary conditions are deduced from the 3-D P_N formulation. The 2-D axisymmetric formulation is implemented in *OpenFOAM* [15] C++ open source libraries. *OpenFOAM* provides the mesh generator, the numerical PDE solvers and the input/output handlers for the example problems shown in this paper. It also includes various CFD calculation modules, which the radiation module can be directly coupled with. Like other modern CFD codes, *OpenFOAM* uses the finite-volume method with unstructured mesh topology. A wedge is the most common way to represent an axisymmetric full cylinder in the finite-volume CFD simulation. Thus, the 2-D axisymmetric example cases in this paper are based on a 3-D finite-volume wedge. Demonstration problems presented here are the 3-D wedge versions of axisymmetric cases presented in [14]. The results of high-order P_N are found to be very close to the exact solution of the RTE, and the results are also verified against those of the 3-D P_N solver.

2. Formulation

Axisymmetric conditions: The radiative transfer equation (RTE) is an integro-differential equation with spatial and directional dependence [1],

$$\hat{s}\cdot\nabla_{\boldsymbol{\tau}} I + I = (1-\omega)I_b + \frac{\omega}{4\pi}\int_{4\pi} I(\hat{s}')\Phi(\hat{s}\cdot\hat{s}')\,d\Omega' \tag{1}$$

where $\boldsymbol{\tau} = \int \beta_r\, d\mathbf{r}$ is an optical coordinate, and β_r is the radiative extinction coefficient; I_b is the blackbody radiative intensity (Planck function); and ω is the scattering albedo. The P_N approximation is based on approximating the radiative intensity field $I(\boldsymbol{\tau},\hat{\mathbf{s}})$ as a series of products of intensity coefficients I_n^m and spherical harmonics Y_n^m, whereby the spatial and the directional ($\hat{\mathbf{s}}$) dependencies are decoupled:

$$I(\boldsymbol{\tau},\hat{\mathbf{s}}) = \sum_{n=0}^{N}\sum_{m=-n}^{n} I_n^m(\boldsymbol{\tau})Y_n^m(\hat{\mathbf{s}}) \tag{2}$$

Spherical harmonics satisfy Laplace's equation in spherical coordinates and are defined here as

$$Y_n^m = \begin{cases} \cos(m\psi)P_n^m(\cos\theta) & \text{for } m \ge 0 \\ \sin(|m|\psi)P_n^m(\cos\theta) & \text{for } m < 0 \end{cases} \tag{3}$$

and $P_n^m(\cos\theta)$ are associated Legendre polynomials. The position-dependent intensity coefficients $I_n^m(\boldsymbol{\tau})$ are determined by applying the series approximation to the RTE.

The radiative intensity depends on position $\mathbf{r}(r,\phi,z)$ and direction $\hat{\mathbf{s}}(\theta,\psi)$ where θ is the polar angle (measured from the z-axis), and ψ is the azimuthal angle (measured counter-clockwise from the x-axis). If the physical system is axisymmetric, then the radiative intensity varies radially with r and axially with z, but not azimuthally with ϕ. Fig. 1 illustrates several location–direction combinations, which have identical intensities for axisymmetric conditions. At a

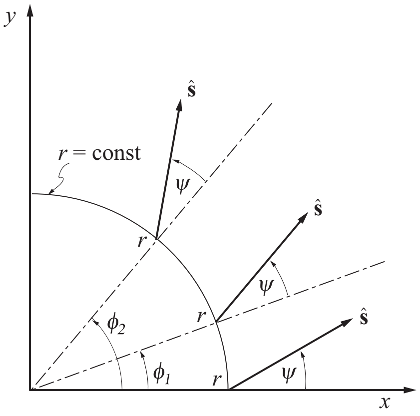

Fig. 1. Illustration of the invariance of intensity with respect to azimuthal angle ψ at different locations for axisymmetric conditions.

fixed location $\mathbf{r}(r,\phi_1,z)$ the radiative intensity in the direction $\hat{\mathbf{s}}(\theta,\psi+\phi_1)$ is equal to the radiative intensity at some other location $\mathbf{r}(r,\phi_2,z)$ in the direction $\hat{\mathbf{s}}(\theta,\psi+\phi_2)$, which has the same deflection angle relative to its position vector $\mathbf{r}$. One may conclude from Fig. 1 that

$$I(r,\phi,z;\theta,\psi+\phi) = I(r,0,z;\theta,\psi) \tag{4}$$

for any ϕ, as long as the problem is axisymmetric. When $\phi=0$, the radiative intensity is evaluated along the x-axis. Considering the general case at some arbitrary ϕ and a reference case when $\phi=0$, the radiative intensity as approximated by the spherical harmonic series expansion equation (2) yields, for a given n, the equality

$$\begin{aligned} &I_n^0(r,\phi,z)P_n^0(\theta) + \sum_{m=1}^{n} I_n^m(r,\phi,z)[\cos m\psi\cos m\phi \\ &\quad -\sin m\psi\sin m\phi]P_n^m(\theta) \\ &\quad +\sum_{m=1}^{n} I_n^{-m}(r,\phi,z)[\sin m\psi\cos m\phi + \cos m\psi\sin m\phi]P_n^m(\theta) \\ &= I_n^0(r,0,z)P_n^0(\theta) + \sum_{m=1}^{n} I_n^m(r,0,z)\cos m\psi P_n^m(\theta) \\ &\quad +\sum_{m=1}^{n} I_n^{-m}(r,0,z)\sin m\psi P_n^m(\theta) \end{aligned} \tag{5}$$

By comparing the I_n^0 terms, it follows that for $m=0$

$$I_n^0(r,\phi,z) = I_n^0(r,0,z) \tag{6}$$

which implies that the intensity coefficients with $m=0$ must be functions of r and z only and are thus axisymmetric. Now comparing other like terms, $\cos m\psi P_n^m(\cos\theta)$ and $\sin m\psi P_n^m(\cos\theta)$ in Eq. (5), yields the following relations for intensity coefficients with $m>0$:

$$I_n^m(r,0,z) = I_n^m(r,\phi,z)\cos m\phi + I_n^{-m}(r,\phi,z)\sin m\phi \tag{7a}$$

$$I_n^{-m}(r,0,z) = -I_n^m(r,\phi,z)\sin m\phi + I_n^{-m}(r,\phi,z)\cos m\phi \tag{7b}$$

Inverting these relations to express $I_n^m(r,\phi,z)$ and $I_n^{-m}(r,\phi,z)$ in terms of the $I_n^m(r,0,z)$ and $I_n^{-m}(r,0,z)$ gives

$$I_n^m(r,\phi,z) = I_n^m(r,0,z)\cos m\phi - I_n^{-m}(r,0,z)\sin m\phi \tag{8a}$$

$$I_n^{-m}(r,\phi,z) = I_n^m(r,0,z)\sin m\phi + I_n^{-m}(r,0,z)\cos m\phi \tag{8b}$$