Journal of Quantitative Spectroscopy & Radiative Transfer 147 (2014) 8–23

Contents lists available at ScienceDirect

Journal of Quantitative Spectroscopy & Radiative Transfer

journal homepage: www.elsevier.com/locate/jqsrt

On radiative transfer using synthetic kernel and simplified spherical harmonics methods in linearly anisotropically scattering media

Zekeriya Altaç *

Department of Mechanical Engineering, School of Engineering and Architecture, Eskisehir Osmangazi University, Batı-Meşelik, 26480 Eskisehir, Turkey

ARTICLE INFO

Article history:
Received 7 February 2014
Received in revised form
3 May 2014
Accepted 5 May 2014
Available online 14 May 2014

Keywords:
Synthetic Kernel method
Anisotropic scattering
Simplified spherical harmonics
Double spherical harmonics
PN approximation

ABSTRACT

The Synthetic Kernel (SK_N) method is employed to a 3D absorbing, emitting and linearly anisotropically scattering inhomogeneous medium. Standard SK_N approximation is applied only to the diffusive components of the radiative transfer equations. An alternative SK_N (SK_N^*) method is also derived in full 3-D generality by extending the approximation to the direct wall contributions. Complete sets of boundary conditions for both SK_N approaches are rigorously obtained. The simplified spherical harmonics (P_{2N-1} or SP_{2N-1}) and simplified double spherical harmonics (DP_{N-1} or SDP_{N-1}) equations for linearly anisotropically scattering homogeneous medium are also derived. Resulting full P_{2N-1} and DP_{N-1} (or SP_{2N-1} and SDP_{N-1}) equations are cast as diagonalized second order coupled diffusion-like equations. By this analysis, it is shown that the SK_N method is a high-order approximation, and simply by the selection of *full* or *half range* Gauss–Legendre quadratures, SK_N^* equations become identical to P_{2N-1} or DP_{N-1} (or SP_{2N-1} or SDP_{N-1}) equations. Numerical verification of all methods presented is carried out using a 1D participating isotropic slab medium. The SK_N method proves to be more accurate than SK_N^* approximation, but it is analytically more involved. It is shown that the SK_N^* with proposed BCs converges with increasing order of approximation, and the BCs are applicable to SP_N or SDP_N methods.

1. Introduction

Radiative heat transfer analysis in a participating medium requires solution of radiative transfer equation (RTE). RTE is an integro-differential equation with five independent variables —three in space (x, y and z) and two in direction (θ and φ) coordinates. This inherent nature of the RTE makes it very difficult to obtain an exact solution. As a consequence, it is not surprising to encounter several approximate methods in the literature to solve the RTE, besides their '*improved*' and/or '*modified*' extensions to overcome certain limitations.

Exact solutions of RTE are obtained from the solution of so called radiative integral transfer equations (RITEs). Since the angular dependency of RTE is completely eliminated, one has to deal with only spatial variables when solving the RITEs. Analytical solutions are available only in 1-D idealized circumstances; on the other hand, numerical solution techniques of RITEs are cpu-time demanding. For this reason, the RITEs are not suitable to solve practical engineering problems. Faced with this reality, analysts have been seeking approximate, yet accurate and efficient methods that have the capability of solving various radiative transfer problems. Most methods in the field of radiative transfer stem from angular

* Tel.: +90 222 2393750; fax: +90 222 2393613.
E-mail address: zaltac@ogu.edu.tr

http://dx.doi.org/10.1016/j.jqsrt.2014.05.008
0022-4073/

Nomenclature

a_1	coefficient of linear anisotropy
D_n	artificial diffusion coefficient defined as $=\mu_n^2/\beta$ (m)
$E_n(x)$	nth order exponential integral function
G	incident energy (W m^{-2})
f_G, $\mathbf{f_q}$	direct radiant energy contributions defined by Eqs. (4) and (5)
I	incident radiation (W m^{-2} sr^{-1})
I_b	blackbody radiation intensity (W m^{-2} sr^{-1})
$K_n(x)$	nth order modified Bessel function
$P_n(x)$	nth order Legendre polynomial
R	radius of cylinder/sphere (m)
$\mathbf{r}$	position vector inside a medium (m)
$\mathbf{r}_w$	position vector on a surface (m)
r_0	radius of a cylinder or sphere (m)
S_0	isotropic medium source defined by Eq. (3)
S_1	anisotropic source defined by Eq. (3)
T	temperature (K)
$\mathbf{q}$	net radiative heat flux (W m^{-2})
w_n	weights of half- or full-range Gauss–Legendre quadratures
x,y,z	coordinates variables
x_0	slab medium half thickness (m)

Greek symbols

β	extinction coefficient (m^{-1})
δ	Delta Dirac function
τ	optical path
κ	absorption coefficient (m^{-1})
ρ	distance between two points
μ	angular cosine
μ_n	abscissas of half- or full-range Gauss–Legendre quadratures
ω	scattering albedo
σ_s	scattering coefficient (m^{-1})

discretization of the RTE. In this paper, however, neither theoretical nor computational aspects of these methods are ventured into; the scope is restricted with the existing methods that do not deal with angular discretization.

Spherical harmonics, or P_N, approximation is perhaps one of the earliest methods employed to the solution of Boltzmann's equation encountered in astrophysics, nuclear reactor physics and thermal radiative transfer. P_1 approximation quickly became attractive due to its simplicity, but soon it was discovered that P_1 (or *diffusion*) approximation was accurate only for optically thick media [1]. Next logical step was to increase the order of approximation to obtain reasonably accurate solutions for optically thin media. As the number of spherical harmonics (order of the P_N) is increased, the accuracy improves only slowly; however, mathematical complexities of the P_N equations increase dramatically [1–4]. This feature of the P_N method became its main shortcoming and an obstacle in incorporating high order approximations into neutron transport (or thermal radiation) calculations.

Another method which found application mostly in neutron transport theory is the so-called double spherical harmonics (or double P_N–DP_N or P_{NN}) approximation introduced by Yvon [5]. In fact, in the field of radiative transfer, a similar approach was independently proposed much earlier by Schuster [6] and Schwarzschild [7]. The method is known as the *Schuster–Schwarzschild* (or *two-flux* or DP_0) *approximation*. The motivation behind the DP_N method is that the P_N method is relatively accurate inside of a large homogeneous medium where intensity is a rather slowly varying function of angle. A discontinuity in intensity at boundaries occurs, and large errors may arise at such points. These are due to the difficulty in expressing a discontinuous function in terms of a series of continuous functions. To improve the convergence of the solution at boundaries, Yvon separates the components of the radiation traveling in the forward and backward directions which yields in improved solutions over the P_N method. The DP_N approximation was generally applied to the radiative transfer of 1-D planar and spherical media [8]. Tsai [9] numerically solved the DP_1 equations in studying combined conduction and radiation heat transfer in absorbing, emitting, and anisotropically scattering layers. Wan et al. [10] used the method for radiative transfer in a planar media with Rayleigh scattering. Mengüç and Iyer [11] used double spherical harmonics method to formulate the DP_1 approximation for a medium with linear-anisotropic scattering. High order DP_N, just as P_N method, has not found widespread applications (other than planar media) in the radiative transfer field basically due to the complexity of the equations.

In light of the foregoing arguments, a simpler alternative of the spherical harmonics (other than P_1) method was sought. With this motivation, about 50 years ago, the simplified P_N (or SP_N) method was introduced by Gelbard [12] to solve the neutron transport equation. Gelbard, taking a heuristic approach, applied the P_N equations of 1-D planar medium to 3-D media, as explained in detail in Section 4. Similar to standard P_N, Mark and Marshak boundary conditions developed for the planar geometry were also employed to the SP_N. Although the SP_N equations are much more easily manageable, the method lacked theoretical foundations which hindered its widespread use in the fields of nuclear reactor physics and thermal radiation. Theoretical justifications for the SP_N were provided by Larsen et al. [13,14] and Pomraning [15]. Larsen used an asymptotic analysis from which he was able to show that the SP_N method was an asymptotic correction to the *diffusion* (*differential*) approximation whereas Pomraning used a variational analysis to asymptotically relate the SP_N to P_N equations in planar geometry. A detailed review of the history and theoretical developments of the SP_N method was given by McClarren [16]. The method, however, found limited applications in the radiation transfer up until recently. Larsen et al. [17] applied the SP_N method to radiative heat transfer in optically thick