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Polarized radiative transfer equation in some curvilinear coordinate systems

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ABSTRACT

The differential operator for polarized radiative transfer equation in homogeneous space obeying Euclidean geometry is explicitly written down for several orthogonal curvilinear coordinate systems of astrophysical interest. The medium is assumed to be statistically homogeneously filled with polydisperse particles, and birefringence in the effective medium is assumed to be negligible.

1. Introduction

Theory of polarized radiative transfer in general, and radiative transfer equation in particular, have a long history of development, gradually including more and more complicated physical models. One of the early versions of the polarized radiative transfer equation was that by Chandrasekhar [1] for Rayleigh scattering in a plane-parallel layer. Not trying to be exhaustive, one can mention the further development e.g. in [2–9]. Polarized radiative transfer in curved spacetime where general relativity is essential was studied in [10–13] and many other papers.

Until now, most part of analytic theory of radiative transfer equation has been elaborated for plane-parallel symmetry of both the medium and the radiation field. Spherical symmetry was considered e.g. in [14–16], while cylindrical symmetry problems were discussed in [17–19]. At the same time, astrophysical objects demonstrate plenty of shapes.

First, let us mention protoplanetary and planetary nebulae, with their very diverse morphology [20,21]. Conical and toroidal structures as well as rotating disks are observed, and rotational ellipsoids (spheroids) are modelled [22]. Sometimes they emit radiation with a very high degree of polarization which in some cases can be mapped over the surface of the object [23]; naturally the integral polarization tends to be low if the object is unresolved [24].

Second, there exist quite many rapidly rotating stars with oblateness up to 20–30% [25]. Naturally, their observable radiation can be polarized. Polarization of such stars was modelled theoretically in [26] assuming that essentially all the radiation comes from photosphere, and for unresolved star it can be calculated integrating the results of plane-parallel radiative transfer approximation over the star's surface. However, rapid rotation (or some other cause) can lead to the formation of a rarified gaseous envelope of nonspherical shape. For example, B5V star BD+31°643 exhibits spatially resolved disk-like structure (one can assume it to be either cylinder, oblate spheroid or something another) with spatially resolved degree of polarization up to 50% [27].

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While numerically modelling the astrophysical objects of sometimes very peculiar shape and morphology, the general theoretical results are used quite rarely – obviously because these last refer only to simplified idealized cases. Researchers do not use analytic solutions but rather construct some numerical grid of spatial coordinates (very often this grid is rectangular) as well as some grid of directions of propagation of radiation somehow distributed over the unit sphere, and let the computers to calculate the results. Nevertheless the numerical solution of any equation of mathematical physics, including the solution of radiative transfer equation, should reflect the analytic properties (first of all, singularities) and asymptotic properties (at very small or very large optical depth; for single scattering albedo close to unity) of the exact solution. Often the above-mentioned shapes of astrophysical objects can be approximated by plane-parallel slabs, spheres, oblate or prolate spheroids, cylinders, cones or toroids. So, the analytic and asymptotic properties of the exact solutions for regular symmetries are desirable to be known. Further, it would be very nice if the numeric computational grid itself reflects the symmetry of the problem. At last, it is desirable to test the computer codes against analytic benchmark solutions if such are known.

While doing the theoretical analysis and derivations, it is desirable to be as general as possible. Many of the coordinate systems reflecting the above-mentioned symmetries can be obtained as some particular or limiting cases of a triaxial ellipsoidal coordinate system, which is a quite general case of coordinate systems with second-order coordinate surfaces. A circular conical coordinate system is the limiting case of elliptic conical coordinate system, and a circular cylindrical coordinate system is a particular case of elliptic cylindrical coordinate system. This leads to the conclusion that all these coordinate systems should be studied at least theoretically, though it is not plausible somebody will model some astrophysical object directly using the theoretical results for triaxial ellipsoidal system.

The first step in this direction is to write down the radiative transfer equation in the curvilinear spatial coordinate system under interest, and of course – accounting for polarization. A general framework how to do it was set in [28] using tensor analysis. While this is enough in principle it can nevertheless be regarded as too big and cumbersome set of mathematical formulae for many people involved in applications and modelling of real objects. It seems justified to uncover the general tensor expressions and to publish a compendium of clear formulae for the set of the most important coordinate systems, in order to avoid repeating one and the same routine calculations in future. This is the aim of the current paper.

Few preliminary results in this direction were published in [29]. Nevertheless, the original parameterization of elliptic conical coordinate system chosen in that paper proved to be not the best possible, and now it has been replaced. Several other results published in the current paper have been presented at two IAU Symposiums and some other conferences as posters but the resulting publications in the respective conference proceedings were essentially thesis, without any equations.

2. Structure of radiative transfer equation in isotropic medium

As it is thoroughly discussed e.g. in [9], the radiative transfer equation itself is an approximation following from Maxwell electrodynamics treated statistically, and it has certain conditions of applicability. In this paper these conditions are assumed to be satisfied.

Let the physical properties of the medium be the same as those assumed in [28,29]:

- The four-dimensional spacetime is flat, obeying Minkowski metrics (general relativity is not essential). Consequently, the three-dimensional space is Euclidean.
- The medium can be filled with absorbing and scattering polydisperse particles. In radiative transfer approximation the polydisperse medium is statistically homogeneous in the whole region under consideration.
- Anisotropy of the polydisperse medium is weak, if it exists at all. The real part of the effective refractive index of polydisperse medium is almost independent of polarization, and there is no significant birefringence.

Let us characterize partially polarized radiation by four real Stokes parameters (I, Q, U, and V) defined as in [9]. One can write a four-dimensional Stokes vector, e.g., in linear polarization (LP) representation,

$$\mathbf{I}(\mathbf{r},\mathbf{\Omega}) = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}(\mathbf{r},\mathbf{\Omega}), \tag{1}$$

or in circular polarization (CP) representation [2,30,9]:

$$\mathbf{I}(\mathbf{r},\mathbf{\Omega}) = \begin{pmatrix} I_2 \\ I_0 \\ I_{-0} \\ I_{-2} \end{pmatrix}(\mathbf{r},\mathbf{\Omega}) = \frac{1}{2}\begin{pmatrix} Q-iU \\ I-V \\ I+V \\ Q+iU \end{pmatrix}(\mathbf{r},\mathbf{\Omega}), \tag{2}$$

where $\mathbf{r}$ is the radius vector of the point of observation, $\mathbf{\Omega}$ is the unit vector in the direction of propagation of radiation, and i is the imaginary unit. (The definition of Stokes parameters (U,V) and the definition of the components of Stokes vector in CP representation differ by signs in different papers written by different authors.) In order to specify Q and U, one needs to