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Modification of the T-matrix method using *a priori* information on singularities of analytical continuation of the wave field

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ABSTRACT

A modification of the T-matrix method is proposed on the basis of the *a priori* information on the analytical properties of the solution. It is shown that the account for such an information allows one to obtain correct, efficient, and universal algorithms in modeling scattering characteristics for a large scale of geometries of the bodies, including essentially non-Rayleigh ones.

1. T-matrix method

In many branches of physics, radiophysics, radioastronomy, and others, a problem of obtaining scattering characteristics averaged over incident angles of the initial wave for different objects is very important. For these purposes, the T-matrix method (TMM) started being applied at the beginning of the 1990s ([1], for instance). The said method was first proposed in [2], and then, 20 years later, has been extensively used ([3] and [4], for instance). The method is based on the well-known null field relation [5]

$$\int_S \left\{ u(\vec{r}\,')\frac{\partial G_0}{\partial n'} - \frac{\partial u(\vec{r}\,')}{\partial n'} G_0(\vec{r},\vec{r}\,') \right\} ds' = \begin{cases} u^1(\vec{r}), & \vec{r} \in \mathbb{R}^n \backslash \overline{D}, \\ -u^0(\vec{r}), & \vec{r} \in D. \end{cases} \quad (1)$$

In this relation, $u = u^0 + u^1$ where u^0 is the initial field (the incident one), u^1 is the secondary (scattered) field, $G_0(\vec{r},\vec{r}\,')$ is the Green function of free space, S is the boundary of the scatterer, and D is the domain inside S. The differentiation in Eq. (1) is made in the direction of the outer normal to D.

First, let us consider a two-dimensional case. Let the impedance boundary condition be at the boundary of the cylinder:

$$\left(u - \frac{Z}{ik\zeta}\frac{\partial u}{\partial n} \right)\Big|_S = 0, \quad (2)$$

where Z is the value of the impedance at the boundary (directrix) of S, with $k = \omega\sqrt{\varepsilon_0\mu_0}$ being the wavenumber and ζ the wave resistance (impedance) in the outer medium.

In the 2D case, $G_0(\vec{r},\vec{r}\,') = (1/4i)H_0^{(2)}(k|\vec{r} - \vec{r}\,'|)$. Let us introduce the following notations:

$$\frac{i}{4}\frac{\partial u}{\partial n'}\Big|_S = \frac{1}{\kappa(\varphi')}J(\varphi'), \quad ds' = \kappa(\varphi')d\varphi', \quad \kappa(\varphi) = \sqrt{\rho^2(\varphi) + \rho'^2(\varphi)}, \quad (3)$$

where $r = \rho(\varphi)$ is the equation describing boundary S.

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With notations (3), boundary condition (2) takes the form

$$u(\vec{r}')\Big|_S = \frac{4W}{ik\kappa(\varphi')} J(\varphi')\Big|_S, \tag{4}$$

where $Z = i\zeta W$, and, for the case at hand, the integral equation for the null field can be written as

$$\int_0^{2\pi} J(\varphi')\left\{H_0^{(2)}(k|\vec{r}-\vec{r}'|) - \frac{W}{k}\frac{\partial H_0^{(2)}(k|\vec{r}-\vec{r}'|)}{\partial n'}\right\}d\varphi' = -u^0(\vec{r}), \quad \vec{r}\in\Sigma, \tag{5}$$

where Σ is some simple curve within S.

The conversion to the T-matrix method is based on choosing a circle with radius $r = r_0$ as Σ, which is wholly covered by S [5].

Publication [3] states that the convergence and accuracy of the T-matrix method substantially worsen as the ratio of the maximal to minimal size of the scatterer increases, this phenomenon being caused by bad conditionality of the matrices. In particular, work [3] argues that the standard T-matrix method is restricted by a value of about 4 of the aforementioned ratio. What is a reason for such restrictions?

Standard algorithms of the T-matrix method ignore an important fact, the localization of the set of singularities of wave field $u^1(\vec{r})$ analytic continuation into domain D [6,7]. Formally, Eq. (1) is valid elsewhere inside D. However, in solving integral Eq. (5) that follows from Eq. (1) with the boundary condition, we use some analytic representations for the desired function $u^1(\vec{r})$. All such representations exist only in the domain outside $\bar{A}$ that is a minimal closed set containing all the singularities of the analytic continuation of the wave field $u^1(\vec{r})$ [6,7]. In particular, representation of a wave field in the form of a row on cylindrical harmonicas converges within domain $r > R_0$ where R_0 is the distance to the farthest point of set $\bar{A}$ from the origin of coordinates [6].

Thus, the traditional algorithm of the T-matrix method is correct if

$$R_0 < \min_{\varphi} \rho(\varphi).$$

Such a condition is meted only by the so-called Rayleigh curves [6,8].

2. Modified T-matrix method

Here we will consider another approach based on the modified null-field method (MNFM) [9,10]. According to this method, contour Σ in Eq. (5) must enclose set $\bar{A}$.

Now, Eq. (5) will be solved by using the method of discrete sources [11]. To that end, the integral in the left-hand side of Eq. (5) is substituted for a sum of point sources localized at S

$$\int_0^{2\pi} J(\varphi')\left\{H_0^{(2)}(k|\vec{r}-\vec{r}'|) - \frac{W}{k}\frac{\partial H_0^{(2)}(k|\vec{r}-\vec{r}'|)}{\partial n'}\right\}d\varphi' \cong \sum_{n=1}^{N} a_n K(\vec{r};\vec{r}_n), \tag{6}$$

where a_n is amplitudes of sources,

$$K(\vec{r};\vec{r}_n) = \left\{H_0^{(2)}(k|\vec{r}-\vec{r}'|) - \frac{W}{k\kappa(\varphi_n)}\left[\rho(\varphi_n)\frac{\partial H_0^{(2)}(k|\vec{r}-\vec{r}'|)}{\partial r'} - \frac{\rho'(\varphi_n)}{\rho(\varphi_n)}\frac{\partial H_0^{(2)}(k|\vec{r}-\vec{r}'|)}{\partial \varphi'}\right]\right\}\Bigg|_{\vec{r}'=\vec{r}_n},$$

Then the left- and right-hand parts of Eq. (5) are equated to each other at the collocation points at Σ. Such a procedure results in the following algebraic system:

$$\sum_{n=1}^{N} a_n K(\vec{r}_m;\vec{r}_n) = -u^0(\vec{r}_m), \tag{7}$$

in which

$$\vec{r}_n = \{\rho(\varphi_n), \varphi_n\}, \quad \vec{r}_m = \{r_\Sigma(\varphi_m), \varphi_m\},$$

where $r_\Sigma(\varphi)$ is the equation of curve Σ in polar coordinates. According to MNFM [9–11], just as in the modified method of auxiliary currents [12], curve Σ is constructed by analytical deforming boundary S of the scatterer up to the singularities of the wave field. The details of such a deformation can be found in publications [9–12].

For the case at hand, the diffraction field, according to Eq. (6), can be represented as a sum of point sources elsewhere out of S:

$$u^1(\vec{r}) \cong \sum_{n=1}^{N} a_n K(\vec{r};\vec{r}_n). \tag{8}$$

Thus system (7) and relation (8) contain the same representations that are correct only in the domain out of set $\bar{A}$ of the wave-field singularities.

In matrix notations, system (7) takes the form

$$K\cdot\bar{a} = \bar{b}, \quad \bar{a} = K^{-1}\cdot\bar{b},$$
$$\bar{a} = \{a_n\}_{n=1}^{N}, \quad \bar{b} = \{b_m\}_{m=1}^{N}, \quad K = \{K_{nm}\}_{n,m=1}^{N},$$
$$b_m = -u^0(\vec{r}_m), \tag{9}$$

where

$$K_{nm} = \left\{H_0^{(2)}(k|\vec{r}_m-\vec{r}_n|) - \frac{W}{k\kappa(\varphi_n)} \times\left[\rho(\varphi_n)\frac{\partial H_0^{(2)}(k|\vec{r}_m-\vec{r}_n|)}{\partial r'} - \frac{\rho'(\varphi_n)}{\rho(\varphi_n)}\frac{\partial H_0^{(2)}(k|\vec{r}_m-\vec{r}_n|)}{\partial\varphi'}\right]\right\}.$$

By using the addition theorem

$$H_0^{(2)}(k|\vec{r}-\vec{r}_n|) = \sum_{p=-\infty}^{\infty} J_p(kr_n)H_p^{(2)}(kr)e^{ip(\varphi-\varphi_n)},$$
$$r > r_1 \equiv \max_{\varphi}\rho(\varphi)$$

we find from Eqs. (8) and (6):

$$u^1(\vec{r}) = \sum_{p=-\infty}^{\infty}\left(\sum_{n=1}^{N} a_n\left\{J_p(kr_n) - \frac{W}{k\kappa(\varphi_n)} \times\left[kr_nJ_p'(kr_n) + ip\frac{\rho'(\varphi_n)}{r_n}J_p(kr_n)\right]\right\}e^{-ip\varphi_n}\right)H_p^{(2)}(kr)e^{ip\varphi} = \sum_{p=-\infty}^{\infty} c_pH_p^{(2)}(kr)e^{ip\varphi}, \tag{10}$$