Journal of Quantitative Spectroscopy & Radiative Transfer 145 (2014) 9–16

Contents lists available at ScienceDirect

Journal of Quantitative Spectroscopy & Radiative Transfer

journal homepage: www.elsevier.com/locate/jqsrt

Reformulation of the M1 model of radiative transfer

Tomoyuki Hanawa [a,b,*], Edouard Audit [a]

[a] Maison de la Simulation, USR 3441, CEA – CNRS – INRIA – University Paris-Sud – University de Versailles, F-91191 Gif-sur-Yvette, France
[b] Center for Frontier Science, Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba 263-8522, Japan

ARTICLE INFO

Article history:
Received 21 September 2013
Received in revised form
18 April 2014
Accepted 20 April 2014
Available online 26 April 2014

Keywords:
Radiative transfer model
M1 model
Hydrodynamics
Relativity

ABSTRACT

We reformulate the M1 model of the radiative transfer, i.e. the moment equations up to the first order, for clarifying the physical and mathematical properties. The M1 model is proved to be equivalent to the hydrodynamic equations of ultra relativistic particles. We show two forms: one expressed with the classical Newtonian velocity and the other expressed with the relativistic four velocity. We use the enthalpy density, i.e. the sum of the energy density and the isotropic component of the pressure, instead of the energy density in both forms. The former serves to show us that the ratio of the flux to the enthalpy density denotes the bulk velocity of the radiation in the M1 model. The latter serves us to propose a Lorentz invariant form useful for taking account of the Doppler shift.

1. Introduction

Radiative transfer is ubiquitous in many scientific fields and therefore a lot of efforts have been dedicated to solving the transfer equation. However, in its original form, the transfer equation depends on 7 independent variables (three for space, one for time, one for the photon frequency and two for their direction of propagation). Due to this large number of degree of freedom, solving the transfer equation in 3D is extremely demanding in terms of computing power and it will be out of reach for radiation-hydrodynamics simulation for the foreseeing future. For this reason, many authors have proposed approximation to the full transfer equation. Many of them consist in taking successive moment of the radiative transfer equation in order to eliminate the angular degrees of freedom [1–3, and references therein]. Using this procedure considerably simplify the transfer equation, but requires a *closure relation* expressing the moment of highest order in terms of moments of lower order to close the system.

The simplest of these moment models is the well known diffusion approximation where only the equation on the 0th moment (i.e. radiative energy) is conserved. The closure relation is obtained by assuming that the photon distribution function (or the pressure tensor) is isotropic. This diffusion approximation is exact in the limit of optically thick media and is rather a strong approximation in other situations. The main drawback of the diffusion approximation is that the radiative flux is always aligned with temperature gradient. There are several ways to go beyond this diffusion approximation, as for example the P_n approximation recently proposed by Schäfer et al. [4] or the spherical harmonic scheme [5,6]. Other authors [2,7–9] have chosen to keep one more equation in the moment hierarchy and to use both the radiative energy and the radiative flux. One then needs a *closure relation* giving the Eddington tensor (i.e. ratio of the pressure tensor and radiative energy) in terms of these two variables. A possible

* Corresponding author at: Center for Frontier Science, Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba 263-8522, Japan.
E-mail addresses: hanawa@cfs.chiba-u.ac.jp (T. Hanawa), edouard.audit@cea.fr (E. Audit).

http://dx.doi.org/10.1016/j.jqsrt.2014.04.014
0022-4073/

way to find this closure is to use the so-called M1 model [2]. The M1 model gives a good approximation to the radiation field in both optically thin and thick regions. A shadow behind an absorber or reflector is often referred to as an evidence. It is also useful to evaluate the interaction of radiation with gas since often the radiative flux as well as the energy density are required for the evaluation. Even if the M1 model is used by several teams [10–16], little work has been dedicated to the study of the physical insight of this model. Indeed, the complexity and non-linearity of the closure relation make our physical insights and numerical analysis difficult. The goal of the present paper is to clarify the nature of the M1 model equations and to propose new variables which express them in a simpler form.

This paper is organized as follows. After a short review of the M1 model equations, we show that they can be expressed in a simpler form if we use the enthalpy density instead of the radiation energy density in Section 2. It is also shown in Section 2 that the ratio of the energy flux to the enthalpy density can be interpreted as the Newtonian bulk velocity of the radiation. In Section 3 we show another form of the M1 model equation in which the four velocity is used instead of the velocity. This second form is proven to be Lorentz invariant. In Section 4 we calculate the wave pattern appearing in the M1 model equations using this new formulation. We show that transverse waves propagate at the normal component of the bulk velocity as in relativistic hydrodynamics. In Section 5 we discuss the Riemann problem of the M1 model equations to show that the M1 model equations are equivalent to the hydrodynamical equations of ultra relativistic particles. We discuss the implications in Section 6. We often omit the source terms, i.e. absorption, emission, and scattering in Section 2 through Section 5 in order to simplify the discussion. The new forms of the M1 model equations are given including the source terms in Appendix A. We discuss the validity for omitting the source terms in the last part of Section 5.

2. M1 model equations

In this section, we briefly recall how the M1 model is obtained from the equation of the radiative transfer. The latter can be written as

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t}+\boldsymbol{n}\cdot\nabla I_\nu = -\kappa_{\nu,a}\,\rho(I_\nu - B_\nu)+\kappa_{\nu,s}\,\rho\left(I_\nu - \frac{1}{4\pi}\int I_\nu\, d\Omega\right), \quad (1)$$

where I_ν and B_ν denote respectively the radiative intensity and the Planck function at the frequency, ν. They are functions of the time, t, position, $\boldsymbol{x}$, the direction of propagation, $\boldsymbol{n}$, and the frequency, ν. The symbols, ρ, c, and $d\Omega$ denote the density, the speed of light, and integration over the solid angle, respectively. In the following we assume for simplicity that the absorption opacity ($\kappa_{\nu,a}$) as well as the scattering ($\kappa_{\nu,s}$) are isotropic and independent of the frequency.

Taking the first two moments of this equation, we obtain

$$\frac{\partial E}{\partial t}+\frac{\partial F_i}{\partial x_i} = -\kappa_a \rho c\left(E - aT^4\right), \quad (2)$$

$$\frac{\partial F_j}{\partial t}+c^2\frac{\partial P_{ij}}{\partial x_i} = -(\kappa_a+\kappa_s)\rho c F_j, \quad (3)$$

where the energy density (E), the energy flux (F_i) and the pressure (P_{ij}) are defined as

$$E = \frac{1}{c}\iint I\, d\Omega\, d\nu, \quad (4)$$

$$F_i = \iint (\boldsymbol{n}\cdot\boldsymbol{e}_i) I\, d\Omega\, d\nu, \quad (5)$$

$$P_{ij} = \frac{1}{c}\iint (\boldsymbol{n}\cdot\boldsymbol{e}_i)(\boldsymbol{n}\cdot\boldsymbol{e}_j) I\, d\Omega\, d\nu, \quad (6)$$

where $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ denote the unit vectors in the i-th and j-th directions, respectively. The symbols, T and a, denote the temperature and the radiation energy density constant, respectively.

System (2) and (3) can then be closed by introducing the M1 closure relation

$$P_{ij} = \left(\frac{3\chi-1}{2}\frac{f_i f_j}{f^2}+\frac{1-\chi}{2}\delta_{ij}\right)E \quad (7)$$

where

$$f_i = \frac{F_i}{cE}, \quad (8)$$

$$f = \sqrt{\sum_{i=1}^{3} f_i^2}, \quad (9)$$

$$\chi = \frac{3+4f^2}{5+2\sqrt{4-3f^2}}, \quad (10)$$

where χ and δ_{ij} denote the Eddington factor and the Kronecker's symbol, respectively.

Eq. (7) can be derived only from the assumption that the photon distribution function is symmetric around the direction parallel to the energy flux. The closure itself, i.e. Eq. (10), was first proposed by Levermore [17] in the context of building flux limited diffusion model. He obtained this relation by assuming that the photon distribution function is restricted to be the Lorentz transform of a Planckian (i.e. that there exists a reference frame where it is isotropic). The same closure relation can be derived from the minimum entropy principle as clearly mentioned in the monograph by Struchtrup [18]. Essentially the same closure relation was obtained for phonon gas hydrodynamics by Larecki [19] and for one dimensional radiative transfer by Fort [20], although the relation to Levermore closure was not mentioned. Dubroca and Feugeas [2] proposed the use of the closure relation to solve the radiative transfer in three dimension.

In addition to these two approaches, Eq. (10) can be derived from another argument. For later convenience, and in analogy with gas thermodynamics, we define the enthalpy density as the sum of the energy density and the isotropic component of the pressure:

$$H = E+P = \frac{3-\chi}{2}E, \quad (11)$$

$$P = \frac{1-\chi}{2}E. \quad (12)$$