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Measuring the void: Theoretical study of scattering by a cylindrical annulus

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ABSTRACT

In this paper, we analyze a monochromatic plane wave scattering from an infinite homogeneous cylindrical annulus. In particular, we study the effect that the inner part of the cylindrical annulus (cylindrical void, if you will) has on the scattered field. This is done by isolating the cylindrical void's contribution to the scattered field. We show that if the cylindrical void is small, then its contribution to the scattered field may be approximated by the "screened cylindrical void" (SCV) approximation. We first develop the SCV approximation in a physically intuitive manner, and then show that it could also be obtained in a more mathematically rigorous manner. Numerical results comparing the SCV approximation to the exact solution are also presented.

1. Introduction

Consider a monochromatic plane wave scattering from an infinitely long homogeneous and isotropic cylindrical annulus with outer radius r_1 and inner radius r_2, see Fig. 1a. Let ϵ_1 denote the permittivity of the space surrounding the cylindrical annulus and let ϵ_2 denote the permittivity of the cylindrical annulus itself, $r_2 < r < r_1$. Let us refer to the region of space inside the cylindrical annulus as the "cylindrical void" and ask what effect the cylindrical void has on the scattered field(s) outside the cylindrical annulus. If one were to *experimentally* investigate this, one would do the following:

(a) measure the total field $V^{(1)}(r,\theta)$ outside the cylindrical annulus ($r > r_1$);
(b) measure the total field $U^{(1)}(r,\theta)$ outside an identical "host cylinder;" i.e., a cylinder of radius r_1 and permittivity ϵ_2, as illustrated in Fig. 1b;
(c) compute the difference between the two fields in (a) and (b):

$$W^{(\mathrm{sca})}(r,\theta) = V^{(1)}(r,\theta) - U^{(1)}(r,\theta). \tag{1}$$

Following the above procedure, $W^{(\mathrm{sca})}(r,\theta)$ contains the effect that the cylindrical void had on the scattered field. In this paper, we show that $W^{(\mathrm{sca})}(r,\theta)$ can be approximated by the scattered field produced by the cylindrical void when a plane wave from a region of space with a permittivity of ϵ_2 is incident on it. This approximation holds if the "screening effect" (discussed in Section 2) of the cylindrical annulus is properly accounted for, and if the cylindrical void is sufficiently small. We refer to this approximation as the *screened cylindrical void* (SCV) approximation. Furthermore, we investigate the rate, denoted by $\mathbb{W}^{\mathrm{ext}}$, at which the energy is extinguished (depleted) by the cylindrical void from the total field outside, $U^{(1)}(r,\theta)$, the host cylinder.

To the best of our knowledge, the SCV approximation and its physical interpretation (see Section 2) has not been

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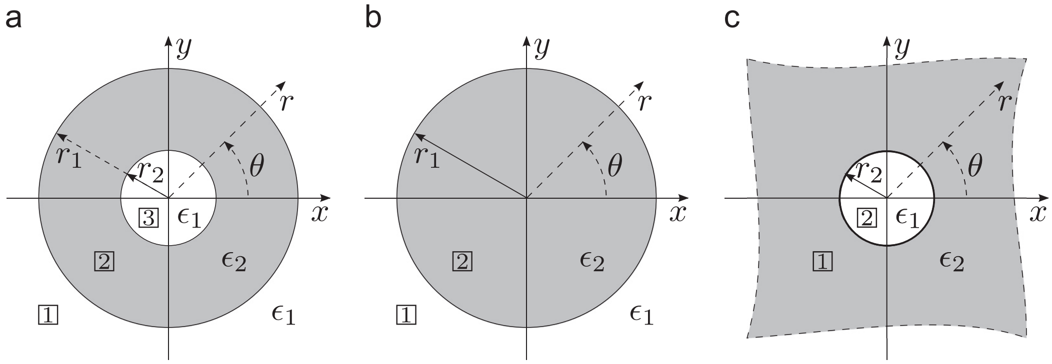

Fig. 1. The cross-sectional view of the cylindrical scattering objects is shown. The origin of the coordinate system (r,θ), where $-\pi \le \theta < \pi$, is concentric with the cylindrical objects. In each panel, the region is denoted by a boxed number and the permittivity of each region is also indicated. For example, region three, $r < r_2$, in panel (a) has a permittivity of ϵ_1 and region one, $r > r_2$, in panel (c) has a permittivity of ϵ_2.

previously considered in the literature. In order to make the paper accessible to the widest possible scientific community, we use the well-known Lorenz–Mie theory [1–4] to derive the SCV approximation. However, we do note that our intuitive derivation of the SCV approximation, which is presented in Section 2, is *physically guided* by the Debye series expansion [5]. In short, the Debye series expansion consists of re-expressing *each* Mie scattering coefficient in terms of an infinite series called the Debye series. Each term in the Debye series may be physically interpreted in terms of the number of reverberations the wave has experienced. A reader interested in the use of the Debye series expansion in the context related to this paper, namely, plane wave scattering by a multilayered cylinder, may consult [6,7] and references therein.

Although we do not explicitly consider many diverse areas of science where the scattering by a cylindrical void is important (e.g., see [3,4]), we would like to mention one, namely, localization. Fifty years after the publication of Anderson's seminal work [8], localization continues to be a thriving area of research [9] in theoretical and experimental physics. Localization of millimeter/submillimeter electromagnetic waves is particularly interesting because both the amplitude and the phase of the electromagnetic field can be easily measured with a vector network analyzer [10]. At these wavelengths, the preparation of disordered samples is also inexpensive and straightforward with standard computer-numerically-controlled (CNC) milling techniques. A sample may be prepared by drilling small holes in a large Teflon (ultra low-loss material) cylinder. Further, by illuminating the sample from the side and putting it on a rotational stage, we can generate essentially arbitrary realizations of the same random disorder. When the number of small scatterers is large, say, over 1000, then what is important is the rate at which the scatterer extinguishes the energy from the incident field, rather than the geometrical shape/size of each individual scatterer [11,12]. Thus, the physical insight into scattering by a single small cylindrical void discussed in this paper may be of benefit in understanding the experimental model described above.

Throughout this paper, we will use the Gaussian unit system, and we will assume that all fields are harmonic in time with a $\exp(-\mathrm{i}\omega t)$ time factor, where ω is the angular frequency. Furthermore, we will assume that all fields are polarized in the positive $\hat{\mathbf{z}}$-direction. The positive $\hat{\mathbf{z}}$-direction is out of the page in Fig. 1. All media considered in this paper are assumed to be non-magnetic, and ϵ_1 is assumed to be purely real.

2. Intuitive derivation of the SCV approximation

In this section, a physically intuitive derivation of the SCV approximation is presented. The derivation is organized as follows. First, we imagine a unit plane wave $u^{(\mathrm{inc})}(r,\theta)$ incident from region one onto the cylindrical void shown in Fig. 1c. Then, we compute the scattered field $u^{(\mathrm{sca})}(r,\theta)$ in region one generated by the scattering of $u^{(\mathrm{inc})}(r,\theta)$ from the cylindrical void. Second, to account for the screening effect of the cylindrical annulus, we use the previously found scattered field $u^{(\mathrm{sca})}(r,\theta)$ as the *incident* (primary) field, i.e., $w^{(\mathrm{inc})}(r,\theta) \equiv u^{(\mathrm{sca})}(r,\theta)$, originating from the center of the host cylinder shown in Fig. 1b. Finally, we compute the total field $w^{(1)}(r,\theta)$ in region one shown in Fig. 1b and physically interpret the terms contained in it to obtain an approximation to $W^{(\mathrm{sca})}(r,\theta)$, see (1).

Let us note that all fields in this paper satisfy the two-dimensional (2D) Helmholtz equation. The radial solution of the 2D Helmholtz equation is composed of a linear combination of integer order Bessel functions of the first and second kind, which we denote by $J_n(\xi)$ and $Y_n(\xi)$, respectively. The Bessel functions $J_n(\xi)$ and $Y_n(\xi)$ also satisfy the Wronskian relationship [13], namely

$$J_{n+1}(\xi)Y_n(\xi) - J_n(\xi)Y_{n+1}(\xi) = \frac{2}{\pi\xi}. \tag{2a}$$

Also, $J_n(\xi)$, $Y_n(\xi)$ and the Hankel function of the first kind, which we denote by $H_n(\xi) = J_n(\xi) + \mathrm{i}Y_n(\xi)$, satisfy the recurrence relation [13]

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Psi_n(\xi) = \frac{n}{\xi}\Psi_n(\xi) - \Psi_{n+1}(\xi), \tag{2b}$$

where Ψ denotes J, Y or H. Lastly, we note the Jacobi–Anger expansion of a plane wave [13], namely,

$$\mathrm{e}^{\mathrm{i}\xi\cos\theta} = \sum_{n=0}^{\infty} g_n \mathrm{i}^n J_n(\xi)\cos(n\theta), \tag{2c}$$