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Angular smoothing and spatial diffusion from the Feynman path integral representation of radiative transfer

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ABSTRACT

The propagation kernel for time dependent radiative transfer is represented by a Feynman path integral (FPI). The FPI is approximately evaluated in the spatial-Fourier domain. Spatial diffusion is exhibited in the kernel when the approximations lead to a Gaussian dependence on the Fourier domain wave vector. The approximations provide an explicit expression for the diffusion matrix. They also provide an asymptotic criterion for the self-consistency of the diffusion approximation. The criterion is weakly violated in the limit of large numbers of scattering lengths. Additional expansion of higher-order terms may resolve whether this weak violation is significant.

1. Introduction

Simulations and imagery of smoke, water, clouds, and other natural phenomena are routinely generated in the computer graphics industry. The imagery is generated primarily from a single scattering approximation, although it can deviate from that for artistic purposes. Multiple scattering is very desirable, but the existing algorithms in use are very ad hoc, not visually good, not sufficiently flexible, and inefficient in the workflow. The algorithm in this paper is the basis for a new software tool addressing all of those limitations (although still with artistic deviations added to it). It has the benefit of beginning with a strong scientific footing in the radiative transfer equation, then is approximated only as much as necessary for the computer graphics application. Also, the algorithm is very flexible for added artistic deviations, and works very well in visual effects production. In this approach, the algorithm should ideally achieve both quasi-ballistic and diffusive regimes of transport behavior, and transition gracefully between the two.

The relationship between radiative transfer and its approximation by diffusion has been of intense interest and application for quite some time. In an extensive review by Davis and Marshak [1], the known methods to relate diffusion and radiative transfer are sorted into two categories. One is a constitutive approach, using Fick's law to relate the scalar flux to vector flux; the other, including asymptotics, is an approximation of radiative transfer in the limit of many scattering events. Both approaches address the extreme of completely diffuse transport, but do not provide a description of how the multiple scattering transitions into diffusion. It would also be very useful to have a framework or model that characterizes the transition from the extreme of near-ballistic behavior to the diffusive regime in scattering media.

In this paper the relationship is analyzed in a new way. The radiative transfer problem is formulated in terms of a Feynman path integral (FPI), which then serves as the starting point for approximations leading to both angular smoothing and spatial diffusion. Radiative transfer has previously been evaluated in terms of FPIs and variations of it [2–5], for applications in ocean optics [6], medical imaging [7], and computer graphics [8,9]. Each application approximated the FPI in accordance with the application's needs, for example using the small angle approximation for ocean and tissue optics. But none of them systematically attempt approximations which could potentially be

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valid in a broad range of applications including diffusive and non-diffusive settings. The approximations in the present paper are less restrictive than those of previous works, and apply across a broader range of problems. In particular, the approximation process used here produces an asymptotic relation for the self-consistency of the procedure.

The approximation is in three steps. The first is an approximation of the functional integral over "momentum," which is defined here as the Fourier conjugate of the increment in direction due to scattering. The result is achieved via a standard stationary phase procedure. The second is approximation of the functional integral over paths through the scattering medium via a steepest descents procedure. These first two steps produce explicit results which are not limited to a diffusive regime. The third and final approximation seeks a simplification that exhibits spatial diffusion. This simplification is the source of the asymptotic relation criterion for diffusion.

The remainder of this paper is organized as follows. In Section 2 the FPI for radiative transfer is examined in preparation for applying approximations. Section 3 discusses the energy conservation property of the radiative transfer propagation kernel, and how that property is preserved in the approximation process. The stationary phase and steepest descents approximations are carried out in Section 4. Spatial diffusion is exhibited in Section 5 after the asymptotic relation is defined and applied to the approximated kernel. The diffusion result is examined for self-consistency with the asymptotic relation as well. Speculation about further research into the mechanism for diffusion is contained in Section 6. Conclusions are presented and discussed in Section 7.

For completeness, the particular version of the FPI used in this paper is derived in Appendix A using the same character of argument used in standard derivations of the FPI in quantum mechanics [10].

Finally, it should be noted that all of the calculations and results for this approximation of the FPI for radiative transfer are shown for the special case of a medium that is uniform with infinite extent. By slightly rephrasing the FPI, a medium with arbitrary spatial variability can be handled. One of the biggest changes in the outcome is that the number of scattering lengths plays a much more important role in sorting out various paths that contribute to the stationary phase and steepest descents approximations. This topic will be the subject of future publications.

2. Feynman path integral representation

The Feynman path integral (FPI) representation applies to the propagation kernel for time dependent radiative transfer. The radiance distribution L satisfies the equations

$$\left\{\frac{\partial}{\partial s}+\hat{\mathbf{n}}\cdot\nabla+c\right\}L(s,\mathbf{x},\hat{\mathbf{n}})=b\int d\Omega(\hat{\mathbf{n}}')P(\hat{\mathbf{n}},\hat{\mathbf{n}}')L(s,\mathbf{x},\hat{\mathbf{n}}')+S(s,\mathbf{x},\hat{\mathbf{n}}),\tag{1}$$

$$L(0,\mathbf{x},\hat{\mathbf{n}})=0,\tag{2}$$

where c and b are the total and scattering extinction coefficients respectively, $a=c-b$ is the absorption coefficient, P is the normalized phase function, and S is a light source. The time s is in units of length after scaling by the light velocity. In terms of a propagation kernel, the explicit integral form of (1) is

$$L(s,\mathbf{x},\hat{\mathbf{n}})=\int_0^s ds'\int d^3x'\int d\Omega(\hat{\mathbf{n}}')G(s-s',\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}}')S(s',\mathbf{x}',\hat{\mathbf{n}}'),\tag{3}$$

where the kernel G satisfies the initial value problem

$$\left\{\frac{\partial}{\partial s}+\hat{\mathbf{n}}\cdot\nabla+c\right\}G(s,\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}}')=b\int d\Omega(\hat{\mathbf{n}}'')P(\hat{\mathbf{n}},\hat{\mathbf{n}}'')G(s,\mathbf{x},\hat{\mathbf{n}}'';\mathbf{x}',\hat{\mathbf{n}}'),\tag{4}$$

$$G(0,\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}})=\delta(\mathbf{x}-\mathbf{x}')\delta(\hat{\mathbf{n}}-\hat{\mathbf{n}}').\tag{5}$$

This integro-differential formulation of the radiative transfer problem also applies to steady-state problems, which can be viewed as a long-time limit of the time-dependent problem. A common application is that the source is always on and time-independent. Taking the long time limit, $s\to\infty$, the time-independent radiance can be expressed with this propagation kernel as

$$L(\mathbf{x},\hat{\mathbf{n}})=\int d^3x'\int d\Omega(\hat{\mathbf{n}}')\ K(\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}}')S(\mathbf{x}',\hat{\mathbf{n}}'),\tag{6}$$

where

$$K(\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}}')=\int_0^\infty ds\ G(s,\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}}').\tag{7}$$

The propagation kernel G has an exact formal expression in terms of a Feynman path integral. This expression is derived in the appendix. The path integral explicitly examines every possible path of length s from the "starting point" $\mathbf{x}'$ and "starting direction" $\hat{\mathbf{n}}'$, which ends at the point $\mathbf{x}$ and direction $\hat{\mathbf{n}}$. Each path is characterized by its unit tangent vector $\hat{\beta}(s')$ at each point $0\le s'\le s$ along the path. Because each path starts at $\mathbf{x}'$ and ends at $\mathbf{x}$, the FPI must only include paths which satisfy

$$\mathbf{x}-\mathbf{x}'=\int_0^s ds'\hat{\beta}(s').\tag{8}$$

Each path contributes a weight factor W in the FPI. The weight is related to how much scattering occurs along the path. The amount of scatter depends on the curvature of the path, $\kappa(s')$, where

$$d\hat{\beta}(s')/ds'=\kappa(s')\hat{\mathbf{N}}(s')\tag{9}$$

and $\hat{\mathbf{N}}$ is the normal to the path in the sense of Frenet–Serret curves. Using the curvature in this way is a choice of convenience to simplify notation. It is not meant to imply that the paths have smooth tangents, and the FPI obtained below does not require them. The scattering weight also depends on the phase function and scattering coefficient. The FPI derivation in the appendix introduces a "momentum" variable $\mathbf{p}(s')$ at each point of the path, which is integrated over all possible momentum configurations. The weight is explicitly expressed as the functional integral[1]

$$W(s,\kappa)=\int[d\mathbf{p}]\exp\left\{i\int_0^s ds'\mathbf{p}(s')\cdot\hat{\mathbf{N}}(s')\kappa(s')\right\}\times\exp\left\{\int_0^s ds' b(\tilde{Z}(|\mathbf{p}(s')|)-1)\right\}\tag{10}$$

[1] Throughout this paper, $[d\cdot]$ denotes a differential element defined with respect to a natural measure; see Appendix A.